Ride the Lightning: The Game Theory of Payment Channels

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Payment channels
Off-chain transaction between neighbors
Off-chain transaction between neighbors
Network creation game
Network creation game

\[
\begin{align*}
 s_0 &= \{3\} \\
 s_1 &= \{3, 4\} \\
 s_2 &= \{3\} \\
 s_3 &= \{} \\
 s_4 &= \{3, 6\} \\
 s_5 &= \{3, 6\} \\
 s_6 &= \{}
\end{align*}
\]
Network creation game

\[
\begin{align*}
    s_0 &= \{3\} \\
    s_1 &= \{3, 4\} \\
    s_2 &= \{3\} \\
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\end{align*}
\]
Network creation game

$s_0 = \{3\}$
$s_1 = \{3, 4\}$
$s_2 = \{3\}$
$s_3 = \{}$
$s_4 = \{3, 6\}$
$s_5 = \{3, 6\}$
$s_6 = \{}$
Network creation game

\[ \text{cost of player 6 under strategy } s = (s_0, s_1, s_2, s_3, s_4, s_5, s_6) \]
Nash equilibrium

A graph is a Nash equilibrium if no player can reduce her cost by unilaterally changing strategy.
The price of anarchy is the ratio of the social costs of the worst-case Nash equilibrium and the social optimum.
Model

\[
cost_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s)
\]

cost of player \(u\) under strategy \(s\)
Players
Channel formation

\[ \text{cost}_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s) \]

number of outgoing channels of player \( u \)
Channel formation

\[
\text{cost}_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s)
\]

**Assumption**: capital considered unlimited.
Channel formation

Assumption: capital considered unlimited.

Assumption: channels initiated unilaterally.

\[
\text{cost}_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s)
\]
Betweenness Centrality

Reflection of the fees a node receives by forwarding the transaction of others.

Closeness Centrality

Measure of the costs encountered for making transactions in the network.
Betweenness Centrality

Closeness Centrality

Reflection of the fees a node receives by forwarding the transaction of others.
Betweenness centrality

\[ \text{cost}_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s) \]

betweenness weight

\[ b \geq 0 \]
Betweenness centrality

$$\text{cost}_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s)$$
Betweenness centrality

\[ \text{cost}_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s) \]
Betweenness centrality

\[ \text{cost}_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s) \]

**Assumption:** uniform transactions.
Betweenness Centrality

Closeness Centrality

Measure of the costs encountered for making transactions in the network.
Closeness centrality

\[
cost_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s)
\]

\(c > 0\)
Closeness centrality

\[ \text{cost}_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s) \]
Closeness centrality

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Closeness centrality

$$\text{cost}_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s)$$

**Assumption:** uniform transactions.
Closeness centrality

\[\text{cost}_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s)\]

**Assumption:** uniform transactions.

**Assumption:** fixed transaction fees.
Social cost

\[
\text{cost}(s) = \sum_{u \in [n]} \text{cost}_u(s)
\]
Social optimum

$$\text{cost}(s) = \sum_{u \in [n]} \text{cost}_u(s)$$

$$\min_s \text{cost}(s)$$
Social optimum

$$\text{cost}(s) = \sum_{u \in [n]} \text{cost}_u(s)$$

$$\min_s \text{cost}(s)$$
Social optimum
Social optimum
Social optimum
Social optimum

\[ \text{cost}(s) = \sum_{u \in [n]} \text{cost}_u(s) \]

\[ \min_s \text{cost}(s) \]
Nash equilibria

- Complete Graph
- Star Graph
- Complete Bipartite Graph
Complete Graph

Star Graph

Complete Bipartite Graph
Complete graph
Complete Graph

Star Graph

Complete Bipartite Graph
Star graph
Complete Graph

Star Graph

Complete Bipartite Graph
Complete bipartite graph

\[ s \geq r \quad r \geq 3 \]
Price of anarchy
Price of anarchy
Price of anarchy
Price of anarchy
Price of anarchy

?
Price of stability
Price of stability
Ride the Lightning:
The Game Theory of Payment Channels
Ride the Lightning:
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tight bound of price of anarchy
Ride the Lightning:
The Game Theory of Payment Channels
Ride the Lightning: The Game Theory of Payment Channels

- Tight bound of price of anarchy
- Modify model for channel initiation
- Weights for players and pairs
Thank You!

Questions & Comments?

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Payment channels network

Layer 2

blockchain
Payment channels network
On-chain channel creation
On-chain channel creation
Off-chain transaction between neighbors
Off-chain transaction between neighbors
Off-chain transaction between neighbors
Off-chain transaction between neighbors
Betweenness centrality

\[ \text{cost}_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s) \]

\[ \text{betweenness}_u(s) = (n - 1)(n - 2) - \sum_{s,r \in [n]: \begin{array}{c} s \neq r \neq u, m(s,r) > 0 \\ m(s,r) \end{array}} \frac{m_u(s,r)}{m(s,r)} \]
Betweenness centrality

\[ \text{cost}_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s) \]

\[ \text{betweenness}_u(s) = \frac{(n - 1)(n - 2)}{\sum_{s,r \in [n]: s \neq r \neq u, m(s,r) > 0} \frac{m_u(s,r)}{m(s,r)}} \]

Ensures positivity of cost function
Betweenness centrality

\[ \text{cost}_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s) \]

\[ \text{betweenness}_u(s) = (n - 1)(n - 2) - \sum_{s, r \in [n]: s \neq r \neq u, m(s, r) > 0} \frac{m_u(s, r)}{m(s, r)} \]
Betweenness centrality

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\text{cost}_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s)
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\text{betweenness}_u(s) = (n - 1)(n - 2) - \sum_{s,r \in [n]: s \neq r \neq u, m(s,r) > 0} \frac{m_u(s,r)}{m(s,r)}
\]
Closeness centrality

\[ \text{cost}_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s) \]

\[ \text{closeness}_u(s) = \sum_{r \in [n]-u} d_{G[s]}(u, r) - 1 \]
Closeness centrality

\[ \text{cost}_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s) \]

\[ \text{closeness}_u(s) = \sum_{r \in [n] - u} d_{G[s]}(u, r) - 1 \]
Social cost

cost(s) = \sum_{u \in [n]} \text{cost}_u(s)

= |E(G)| + b \sum_{u \in [n]} \text{betweenness}_u(s) + c \sum_{u \in [n]} \text{closeness}_u(s)

= |E(G)| + b \cdot n \cdot (n - 1)(n - 2) + (c - b) \cdot \sum_{u \in [n]} \text{closeness}_u(s)
Social cost

\[
\text{cost}(s) = \sum_{u \in [n]} \text{cost}_u(s)
\]

\[= |E(G)| + b \sum_{u \in [n]} \text{betweenness}_u(s) + c \sum_{u \in [n]} \text{closeness}_u(s)\]

\[= |E(G)| + b \cdot n \cdot (n - 1)(n - 2) + (c - b) \cdot \sum_{u \in [n]} \text{closeness}_u(s)\]

\[\star \overline{B}(G) = (n - 1)(\overline{l}(G') - 1)\]

\(\overline{B}(G)\) average betweenness

\(\overline{l}(G)\) average distance
Social optimum \((b \leq c)\)

\[
\text{cost}(s) = |E(G)| + b \cdot n \cdot (n - 1)(n - 2) + (c - b) \sum_{u \in [n]} \sum_{r \in [n] - u} (d_{G[s]}(u, r) - 1) \\
\geq |E(G)| + b \cdot n \cdot (n - 1)(n - 2) + (c - b)(n \cdot (n - 1) - 2|E|) \\
= (1 - 2 \cdot (c - b)) \cdot |E(G)| + b \cdot n \cdot (n - 1)(n - 2) + (c - b)(n \cdot (n - 1))
\]
Social optimum \( b \leq c \)

\[
\text{cost}(s) = |E(G)| + b \cdot n \cdot (n - 1)(n - 2) + (c - b) \sum_{u \in [n]} \sum_{r \in [n] - u} (d_{G[s]}(u, r) - 1) \\
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= (1 - 2 \cdot (c - b)) \cdot |E(G)| + b \cdot n \cdot (n - 1)(n - 2) + (c - b)(n \cdot (n - 1))
\]

\( \star \) all nodes that are not connected by an edge are at least distance two apart
Social optimum \((b \leq c)\)

\[
\text{cost}(s) = |E(G)| + b \cdot n \cdot (n-1)(n-2) + (c-b) \sum_{u \in [n]} \sum_{r \in [n]-u} (d_{G[s]}(u,r) - 1)
\]

\[
\geq |E(G)| + b \cdot n \cdot (n-1)(n-2) + (c-b)(n \cdot (n-1) - 2|E|)
\]

\[
=(1 - 2 \cdot (c-b)) \cdot |E(G)| + b \cdot n \cdot (n-1)(n-2) + (c-b)(n \cdot (n-1))
\]

\[
c > \frac{1}{2} + b \quad \text{complete graph}
\]

\[
b \leq c \leq \frac{1}{2} + b \quad \text{star graph}
\]
Social optimum \((b > c)\)

\[
\text{cost}(s) = |E(G)| + b \cdot n \cdot (n-1)(n-2) - (b-c) \cdot \sum_{u \in [n]} \sum_{r \in [n]-u} (d_{G[s]}(u,r) - 1)
\]

\[
= |E(G)| - 2 \cdot (b-c) \cdot d(G) + b \cdot n \cdot (n-1)(n-2) + (b-c) \cdot n \cdot (n-1)
\]

\[
\geq \left( 1 + b \cdot n \cdot (n-2) + \frac{b-c}{3} n \cdot (n-2) \right) (n-1)
\]
Social optimum \((b > c)\)

\[
\text{cost}(s) = |E(G)| + b \cdot n \cdot (n-1)(n-2) - (b-c) \cdot \sum_{u \in [n]} \sum_{r \in [n] - u} (d_{G[s]}(u, r) - 1)
\]

\[
= |E(G)| - 2 \cdot (b-c) \cdot d(G) + b \cdot n \cdot (n-1)(n-2) + (b-c) \cdot n \cdot (n-1)
\]

\[
\geq \left(1 + b \cdot n \cdot (n-2) + \frac{b-c}{3} n \cdot (n-2)\right) (n-1)
\]

\star \text{ path graph}
NP-hardness

\[ b = 0 \]
\[ 0.5 < c < 1 \]
NP-hardness

\[ b = 0 \]
\[ 0.5 < c < 1 \]
NP-hardness

$\begin{align*}
b &= 0 \\
0.5 < c < 1
\end{align*}$
Complete graph
Complete graph

\[ \Delta \text{cost} = 1 - c \]
Complete graph
Path graph
Path graph

- $n = 4$
- $n = 5$
- $n = 6$
- $n = 7$

- theory
- Nash equilibrium
- no Nash equilibrium
Path graph

\[ \Delta \text{cost}_u(s \text{ to } \tilde{s}) = -c \cdot (m - 2) \]
Path graph
Path graph
Circle graph
Circle graph

- Theory
- Nash equilibrium
- No Nash equilibrium
Circle graph

- $n = 4$
- $n = 5$
- $n = 6$
- $n = 7$

- theory
- Nash equilibrium
- no Nash equilibrium
Circle graph

betweenness_0(s) = \frac{3}{4} \cdot n^2 + o(n^2)

closeness_0(s) = \frac{1}{4} \cdot n^2 + o(n^2)
Circle graph

betweenness_0(\tilde{s}) = \frac{11}{16} \cdot n^2 + o\left(n^2\right)

closeness_0(\tilde{s}) = \frac{3}{16} \cdot n^2 + o\left(n^2\right)
Circle graph

\[ \Delta \text{cost}_u(s \text{ to } \tilde{s}) = - \left( \frac{1}{16} n^2 + o(n^2) \right) (b + c) \]
Circle graph
Circle graph
Star graph
Star graph

\[ \Delta \text{cost} = n - 2 - \frac{(n - 2) \cdot (n - 3)}{2} b - (n - 2) \cdot c \]

\[ 0 \leq 1 - \frac{n - 3}{2} b - c \]
Star graph

$n = 4$

$n = 7$
Complete bipartite graph

$s \geq r$

$r \geq 3$
Complete bipartite graph

\[ r \geq 3 \]

\[ s \geq r \]
Complete bipartite graph
Complete bipartite graph

\[ \Delta \text{cost}_u(s \text{ to } \tilde{s}_1) = -(s - 1) + \frac{s \cdot (s - 1)}{r} b + (s + r - 3) \cdot c \]

\[ 1 \leq \frac{s}{r} b + \frac{s + r - 3}{s - 1} c \]
Complete bipartite graph

$$\Delta \text{cost}_u(s \text{ to } \tilde{s}_2) = 2 - s + \left(\frac{s \cdot (s - 1)}{r}\right) b + (s - 2) \cdot c$$

$$1 \leq \left(\frac{s \cdot (s - 1)}{r \cdot (s - 2)}\right) b + c$$
\[
\Delta \text{cost}_u(s \text{ to } \tilde{s}_3) = r - s + 1 + \left( \frac{s \cdot (s - 1)}{r} - \frac{(r - 1)(r - 2)}{s + 1} \right) b + (s - r + 1) \cdot c
\]

\[
1 \leq \frac{1}{(s - r + 1)} \left( \frac{s \cdot (s - 1)}{r} - \frac{(r - 1)(r - 2)}{s + 1} \right) b + c
\]
Complete bipartite graph
Price of anarchy ($c > 1$)

c > \frac{1}{2} + b

\[
\rho(G) = \frac{\text{cost}(\text{complete graph})}{\text{cost}(\text{complete graph})} = \mathcal{O}(1)
\]
Price of anarchy \((c > 1)\)

\[
c > \frac{1}{2} + b
\]

\[
\rho(G) = \frac{\text{cost(complete graph)}}{\text{cost(complete graph)}} = O(1)
\]

\[
b \leq c \leq \frac{1}{2} + b
\]

\[
\rho(G) = \frac{\text{cost(complete graph)}}{\text{cost(star graph)}} = \frac{\left(\frac{1}{2} + (n - 2) \cdot b\right) \cdot n}{1 + (c + b \cdot (n - 1)) (n - 2)} = O(1)
\]
Price of anarchy ($c > 1$)

$c > \frac{1}{2} + b$

\[
\rho(G) = \frac{\text{cost(complete graph)}}{\text{cost(complete graph)}} = \mathcal{O}(1)
\]

$b \leq c \leq \frac{1}{2} + b$

\[
\rho(G) = \frac{\text{cost(complete graph)}}{\text{cost(star graph)}} = \frac{\left(\frac{1}{2} + (n - 2) \cdot b\right) \cdot n}{1 + (c + b \cdot (n - 1))(n - 2)} = \mathcal{O}(1)
\]

$c < b$

\[
\rho(G) = \frac{\text{cost(complete graph)}}{\text{cost(path graph)}} = \frac{\left(\frac{1}{2} + (n - 2) \cdot b\right) \cdot n}{1 + \left(\frac{2}{3}b - \frac{1}{3}c\right) \cdot n \cdot (n - 2)} = \mathcal{O}(1)
\]
Price of anarchy ($c + b \leq 1/n^2$)

$$\Delta \text{cost}_u(s) > -n^2 \cdot c - n^2 \cdot b + 1$$

spanning trees

$$\text{cost}(s) = \Theta(n)$$

$$\rho(G') = \mathcal{O}(1)$$
Price of anarchy \((c \leq 1 \& c + b \geq 1/n^2)\)

\[
\rho(G) = \mathcal{O}\left( \frac{|E(G)| + n^3 \cdot b + (c - b) \cdot \sum_{u \in [n]} \sum_{r \in [n]-u} (d_G(u, r) - 1)}{n^3 \cdot b + n} \right)
\]
Price of anarchy \((c \leq 1 \& c + b \geq 1/n^2)\)

\[
\rho(G') = \Theta \left( \frac{|E(G')| + n^3 \cdot b + (c - b) \cdot \sum_{u \in [n]} \sum_{r \in [n] - u} (d_G(u, r) - 1)}{n^3 \cdot b + n} \right)
\]

\[
d_G(u, r) < \Theta \left( \frac{2}{\sqrt{c + b}} \right)
\]

\[
\rho(G') = \Theta \left( \frac{|E(G')| + n^3 \cdot b + n^2 \frac{c - b}{\sqrt{b + c}}}{b \cdot n^3 + n} \right)
\]
Price of anarchy \((c \leq 1 \& c + b \geq 1/n^2)\)

\[ \rho(G') = \mathcal{O} \left( \frac{|E(G)| + n^3 \cdot b + (c - b) \cdot \sum_{u \in [n]} \sum_{r \in [n]-u} (d_G(u, r) - 1)}{n^3 \cdot b + n} \right) \]

\[ \mathcal{O} \left( \frac{n^3 \cdot b}{n^3 \cdot b + n} \right) = \mathcal{O}(1) \]

\[ \mathcal{O} \left( \frac{n^2 \cdot \frac{c-b}{\sqrt{b+c}}}{n^3 \cdot b + n} \right) = \mathcal{O} \left( \frac{c-b}{n^2 \cdot b + 1} \right) = \mathcal{O}(1) \]

\[ \mathcal{O} \left( \frac{|E(G)|}{b \cdot n^3 + n} \right) = \mathcal{O}(n) \]

\[ \rho(G') = \mathcal{O}(n) \]