Directed Graph Exploration

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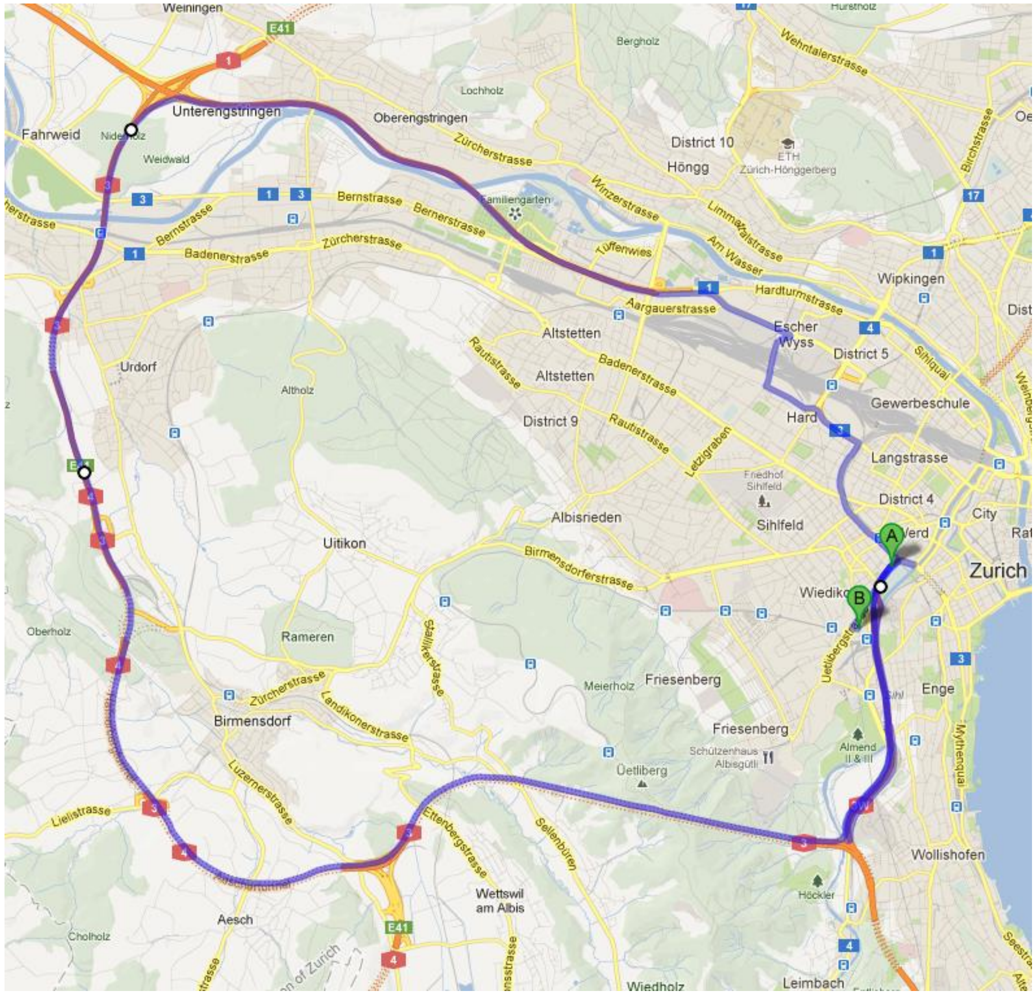
When in Rome …

Navigating in Zurich

Zurich: Full of one-way streets…



Formal Model

- Given a strongly connected directed graph $G = (V, E)$
  - All m edges have non-negative weights
  - All n nodes have a unique ID

- A searcher starts from some node s
  - With unlimited memory and computational power
  - Has to explore the graph

- A graph is called explored, if the searcher has visited all n nodes and returned to the starting node s

- When the searcher arrives at a node, she knows all outgoing edges, including their cost and the ID of the node at the end of the edges

cf. [Kalyanasundaram & Pruhs 1994, Megow et. al. 2011]

How good is a tour, how good is a strategy?

- Cost of a tour: Sum of traversed edge weights

Competitive ratios for:

- a tour T: $$\frac{cost\ of\ T}{cost\ of\ optimal\ tour}$$
- deterministic algorithms: $$\max_{\forall tours\ T} \frac{cost\ of\ T}{cost\ of\ optimal\ tour}$$
- randomized algorithms: $$\max_{\forall tours\ T} \frac{expected\ cost\ of\ T}{cost\ of\ optimal\ tour}$$

Applications of Graph Exploration

- One of the fundamental problems of robotics
  cf. [Burgard et. al. 2000, Fleischer & Trippen 2005]

- Exploring the state space of a finite automaton
  cf. [Brass et. al. 2009]

- A model for learning
  cf. [Deng & Papadimitriou 1999]

Some Related Work

- Offline: Asymmetric Traveling Salesman problem
  - Approximation ratio of $\frac{2}{3}\log_2 n$ [Feige & Singh 2007]

Undirected graph exploration:

Directed Case

- General case: $O(\log n)$ [Rosenkrantz et. al. 1977]
  - Best known lower bound: $2.5 - \varepsilon$ [Dobrev et. al. 2012]
- Planar graphs: 16 [Kalyanasundaram & Pruhs 1994]

$\Theta(n)$

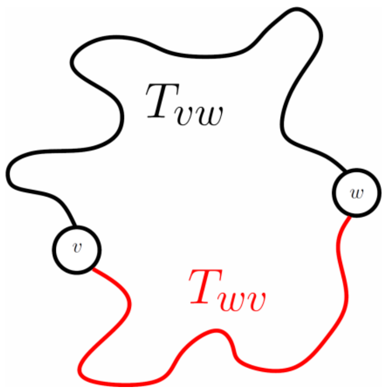
- Genus at most g : $16(1 + 2g)$ [Megow et. al. 2011]
- Unweighted: 2 $(\text{l.b.}: 2 - \varepsilon$, [Miyazaki et. al. 2009])

- Does randomization help?

factor of 4 at most

Exploring with a Greedy Algorithm

- Achieves a competitive ratio of $\boldsymbol{n-1}$
- Proof sketch:
  - Greedy uses $n-1$ paths to new nodes and then returns
  - The greedy path P_{vw} from v to a not yet visited node w is a shortest path
  - Let T be an opt. Tour inducing a cyclic ordering of all n nodes in G, with the tour consisting of n segments.
  - The path P_{vw} has by definition at most the cost of the whole part T_{vw} of the tour T, which consists of at most $n-1$ segments.
  - Therefore, the cost of each of the n segments in T has to be used at most $n-1$ times for the upper cost bound of the greedy algorithm.



Exploring with a Greedy Algorithm – Unweighted Case

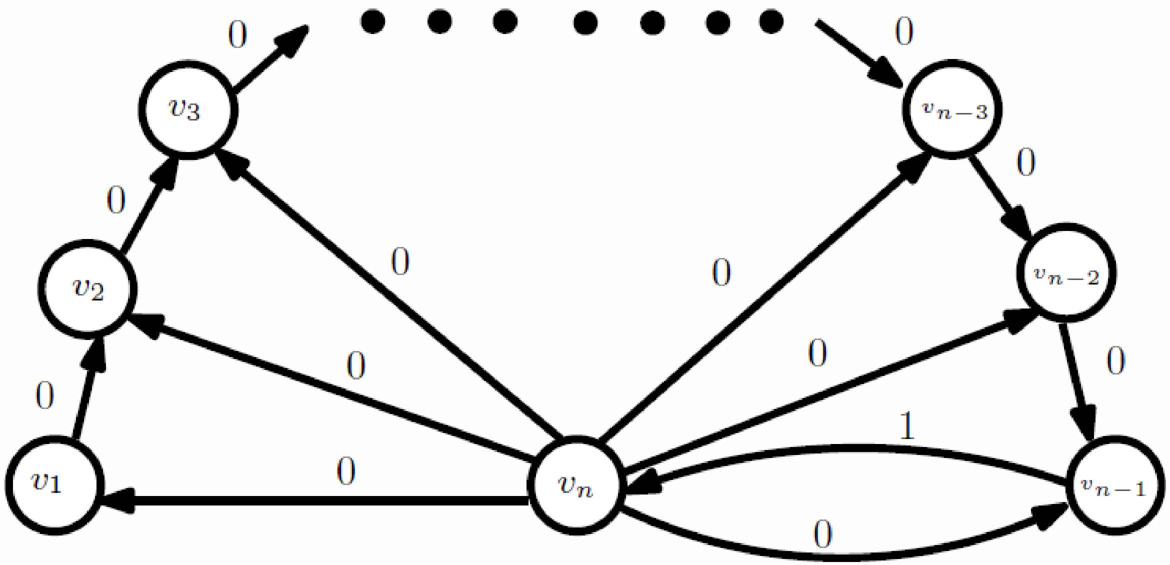
- Achieves a competitive ratio of $\boldsymbol{\frac{n}{2} + \frac{1}{2} - \frac{1}{n}}$

- Proof sketch:
  - The cost to reach the first new node is 1, then at most 2, then at most 3, …
  - If we sum this up, we get an upper bound of

$$1 + 2 + 3 \ldots + (n-2) + (n-1) + (n-1)$$
$$= -1 + \sum_{i=1}^{n} i = \frac{n^2}{2} + \frac{n}{2} - 1$$

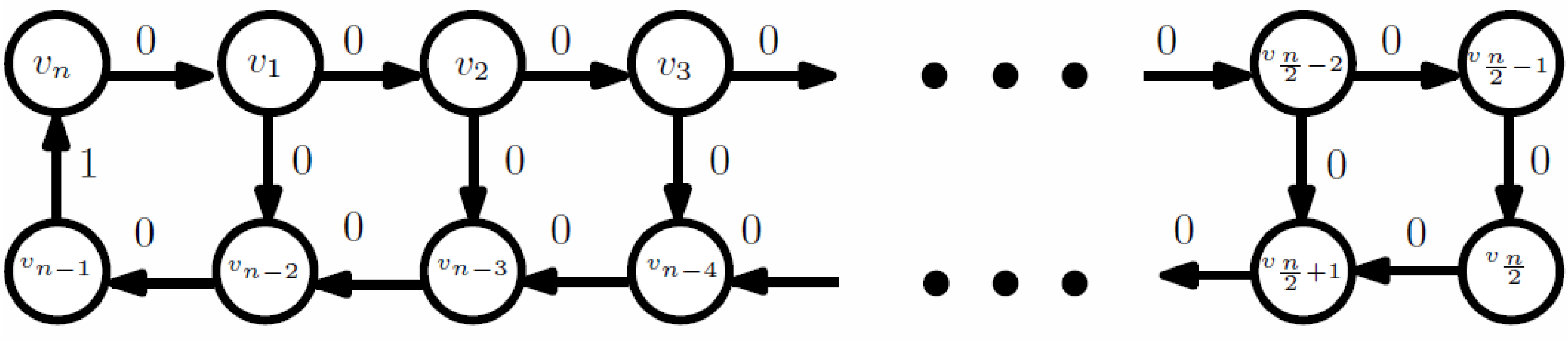
  - The cost of an optimal tour is at least n.

Lower Bounds for Deterministic Online Algorithms



- No better competitive ratio than $n-1$ is possible.
- Unweighted case: No better competitive ratio than $\frac{n}{2}+\frac{1}{2}-\frac{1}{n}$ is possible.
- Both results are **tight**.

Lower Bounds for Randomized Online Algorithms



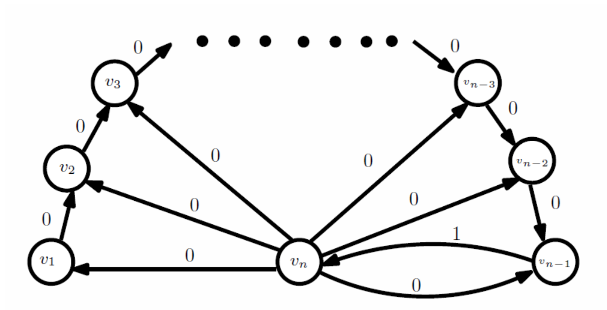
- No better competitive ratio than $\frac{n}{4}$ is possible.

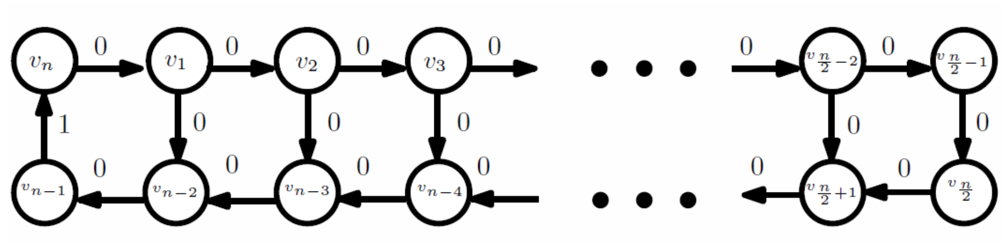
- Proof sketch:
  - When being at a node v_i , with $1 \leq i \leq \frac{n}{2} - 2$, for the first time, then the "correct" edge can be picked with a probability of at most $p = 0.5$.
  - Expected amount of "wrong" decisions: $0.5\left(\frac{n}{2} - 2\right) = \frac{n}{4} - 1$.
  - The cost of an optimal tour is 1.

- Unweighted case: No better competitive ratio than $\frac{n}{8} + \frac{3}{4} - \frac{1}{n}$ is possible.

Variations of the Model

- Randomized starting node?
- Choosing best result from all starting nodes?



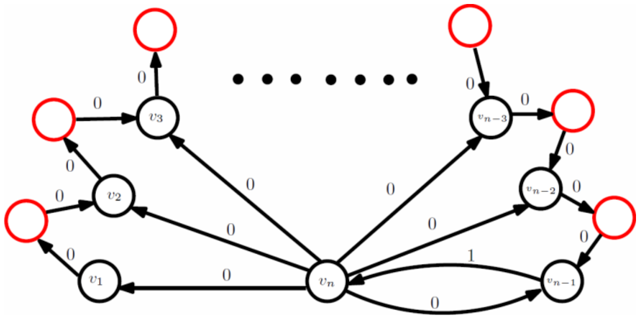


- Possible solution: Duplicate the graphs, connect their starting nodes

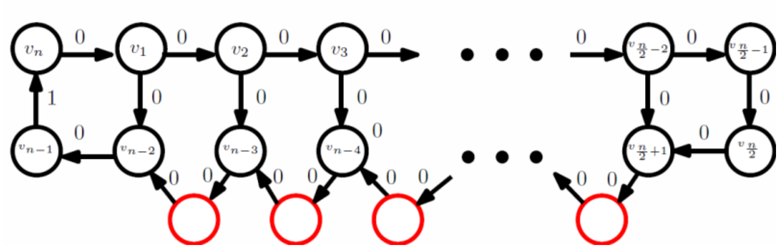
- No better competitive ratio possible than
  - $\frac{n}{4}$ (deterministic online algorithms)
  - $\frac{n}{16}$ (randomized online algorithms)

Variations of the Model

- What if the searcher also sees incoming edges?



decreases lower bound
by a factor of less than 2

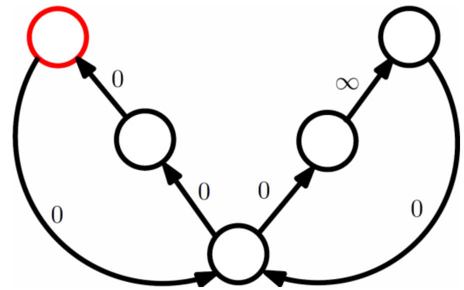


decreases lower bound
by a factor of less than 1.5

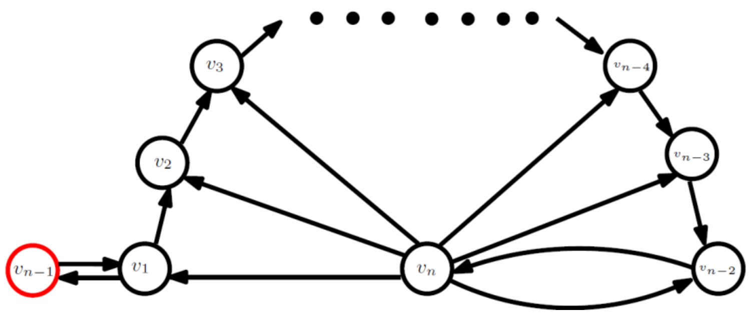
- What if the searcher does not see the IDs of the nodes at the end of outgoing edges, but knows the IDs of outgoing and incoming edges?
  - Greedy algorithm still works with same ratio (all nodes have been visited if all edges have been seen as incoming and outgoing edges)
  - Lower bound examples also still work

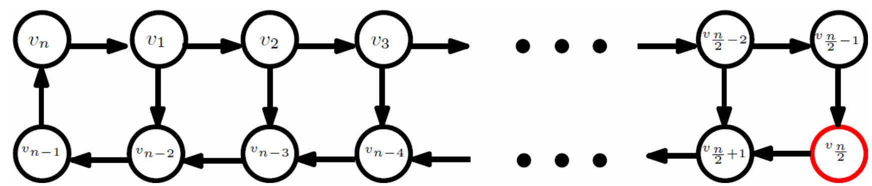
Searching for a Node

- Not feasible in weighted graphs:



- In unweighted graphs, lower bounds for competitive ratios:





Deterministic

$$\frac{(n-1)^2}{4} - \frac{(n-1)}{4} - \frac{1}{2} \in \Omega(n^2)$$

Randomized

$$\frac{(n-1)}{4} + \frac{2}{(n-1)} + \frac{1}{2} \in \Omega(n)$$

- A greedy algorithm has a competitive ratio of $\frac{n^2}{4} - \frac{n}{4} \in O(n^2)$

Overview of our Results

competitivity / type of graph	lower bound	upper bound	multiplicative gap
(deterministic) general*c	$n-1$	$n-1$	sharp
(randomized) general*+c	$\frac{n}{4}$	$n-1$	≤ 4
(determ.) unweighted general*	$\frac{n}{2}+\frac{1}{2}-\frac{1}{n}$	$\frac{n}{2}+\frac{1}{2}-\frac{1}{n}$	sharp
(random.) unweighted general*	$\frac{n}{8}+\frac{3}{4}-\frac{1}{n}$	$\frac{n}{2}+\frac{1}{2}-\frac{1}{n}$	≤ 4
(deterministic) euclidean planar	$n-2-\bar{\epsilon}$	$n-1$	$\leq 1.25+\epsilon$
(randomized) euclidean planar	$\frac{n}{4}-\bar{\epsilon}$	$n-1$	$\leq 4+\epsilon$
(d.) unit weight euclidean planar	$\frac{n}{4}+\frac{1}{2}-\frac{2}{n}$	$\frac{n}{2}+\frac{1}{2}-\frac{1}{n}$	≤ 2
(r.) unit weight euclidean planar	$\frac{n}{8}+\frac{3}{4}-\frac{1}{n}$	$\frac{n}{2}+\frac{1}{2}-\frac{1}{n}$	≤ 4

* also applies to planar graphs and graphs that satisfy the triangle inequality

c also applies to complete graphs and graphs with any diameter from 1 to $n-1$

+ also applies to graphs with any maximum incoming/outgoing degree from 2 to $n-1$ and to graphs with any minimum incoming/outgoing degree from 1 to $n-1$

Thank you

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