What is the Price for Lending in Financial Networks?

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Abstract. We explore the impact of the network structure on lending in financial networks. Specifically, we ask what price agents should charge when lending to each other and how much this price should depend on the network topology. We begin with a simple model that incorporates debts and external assets. We then extend this model to account for default costs as well. In the simplest model, all agents have an identical break-even price and all agents offering this price is a Nash equilibrium. This price is $\frac{1}{r_b}$ per unit of money, where r_b is the recovery rate of the borrower. In the extended model, with default costs, agents can have different break-even prices. Yet surprisingly, we find that a Nash equilibrium is for all agents to bid the same price of $\frac{L_b}{r_b L_b - (1-\alpha)d}$, where L_b is the total debt of the borrower, $(1-\alpha)$ is the default loss rate, and d is the amount being borrowed. Our study shows that network location does not impact debt pricing. However, the overall network topology plays a crucial role in the stability of the network regarding new debt contracts. We show that a single debt contract can push almost every agent in a network into default. Moreover, requiring agents to hold a fixed fraction of their liabilities in liquid assets does not rule out such scenarios.

Keywords: Financial networks · Lending · Lending Games · Multi-Agents.

1 Introduction

Lending is a fundamental aspect of banking and is well-studied in the financial literature. The price of borrowing depends on several factors, most importantly, the creditworthiness of the borrower. Borrowers are often viewed in isolation; however, financial institutions are invariably linked in complex networks or graphs, with nodes representing institutions and directed edges representing liabilities. An unpaid debt in one place can start a chain reaction, ultimately leading to bankruptcies far away from the root cause [27]. Therefore, one may wonder whether network effects also influence lending. Does it matter how the borrower and the lender are related in the network? Surprisingly, this question has been largely overlooked in prior studies. In this paper, we explore the impact of network effects on loan pricing. Please, see fig. 1 for a motivating example.

We consider lending in the standard financial network model with and without default costs [8]. Interestingly, we find that without default costs, all agents

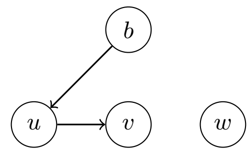

Fig. 1: Agent b wants to borrow some money, and agents u, v, and w are potential lenders. However, their situations are different. Agent b already owes u money; v's value might be dependent on the loan from b to u being paid in full and w is so far completely independent of b. It seems intuitively reasonable that u, v, and w can all charge a different price for the same loan to b.

in the network have the same break-even price, regardless of their exposure to the borrower. In other words, it does not matter who the borrower already owes money to, all agents will break even (make a non-negative profit from the loan) at the same price. When lending to a specific agent b, this price is given by $\frac{1}{r_b}$ per unit lent, where r_b is the recovery rate of the borrower. The recovery rate r_b is the proportion of its debts that b can pay off. It can be seen as a measure of the creditworthiness of b. If agent b with recovery rate r_b wants to borrow d, then the return liability for the borrowed money will be $\frac{d}{r_b}$. If the lender also wants to make a profit, then they will have to add this to the price they offer.

Moreover, we show that this break-even price is also an equilibrium price for all agents, i.e., all agents bidding the break-even price is a Nash equilibrium when considering lending as a game. In particular, agents u, v, and w in fig. 1 would all offer b the same price for a loan. The equilibrium price is computed using a three-way comparison between market clearing before the new debt contract, market clearing after the agent under consideration wins the debt contract, and market clearing after someone else wins the contract.

In the extended model with default costs, different agents can have different break-even prices. In particular, an agent with direct exposure to a bankrupt agent b will need to charge a higher price for an additional loan to b to break even. As such, one would expect that an isolated agent, with the lowest break-even price, would be willing to offer the lowest price for a new loan. Remarkably, however, we find that there is a universal equilibrium price for lending. When one takes into account that another agent could win the contract, an exposed agent is willing to offer below its break-even price even though this results in an overall loss.

Finally, we look more closely at what happens to the network when a new debt contract is introduced. Specifically, we analyze the stability of financial networks to new contracts. We prove that in some networks the addition of a single loan can bankrupt (send into default) almost every agent. Moreover, mandating agents to maintain a fixed fraction of their liabilities in liquid assets, as required by international banking standards, does not rule out such scenarios.

2 Related Work

Debt Pricing Asset and debt pricing is well-studied in the literature, as is evident from the books by Bluhm et al. [6], McNeil et al. [21], and Lando [20]. However, it should be noted that these works disregard the influence of network effects. Typically, these models are constructed based on the computation of anticipated discounted future cash flows. They incorporate the concept of the time value of money, which can vary depending on the lender, along with a risk premium determined by the borrower. Furthermore, these models take into account other potential risks associated with the non-materialization of future cash flows.

The seminal work of Merton [22] applies Black-Scholes-type pricing to corporate debt, taking into account the stochastic changes in the probability of default of the underlying corporation. This probability of default is modeled as a Brownian motion and does not take the probability of default of other corporations into account. In contrast, we focus on the impact of the network on pricing.

Financial Networks Financial networks were introduced in the pioneering works of Allen et al. [1] and Eisenberg and Noe [8]. Eisenberg and Noe's model assumes external assets and simple debt contracts between agents. It has since been extended to include default costs [26], cross-ownership [10, 30], fire sales [7], credit default swaps [28, 24, 25, 19], and debts due in time intervals [12]. A focal topic in the field of financial networks is financial contagion [14, 15, 4]. These works study how an initial shock spreads through the network based on the topology. In particular, they explore what network structures lead to the greatest amplification of shocks and what regulations should therefore be put in place to avoid such scenarios. For a survey, see [16, 18].

Another line of work looks at what central agents can do to prevent or minimize the effects of financial contagion. Freixas et al. [11] and Papachristou et al. [23] examine how liquidity shocks can be compensated for by a central agent. Bernard et al. and Hoefer et al. [5, 17] take a game-theoretic perspective to analyze when agents are incentivized to bail each other out rather than relying on a central agent.

As a consequence of such research, agents and regulatory bodies have acknowledged the importance of taking network effects into account when developing stress testing models [2, 9, 13, 31]. However, based on an empirical study, Siebenbrunner et al. [29] find that financial contagion is still not priced into interbank markets.

Debt Pricing in Financial Networks In contrast to these works, we are interested in the day-to-day operations of agents and how these are affected by the network topology. We study how market values change when a new debt contract is made. Based on this, we calculate the correct price for lending to a specific agent in the network. Banerjee et al. [3] address the issue of pricing existing debt in a financial network with interbank liabilities and comonotonic endowments. In contrast to their work, we focus on pricing new debt contracts in a competitive game-theoretic setting.

3 Preliminaries

We consider a network with n agents, $N := \{u, v, w, \ldots\}$. Each agent may have liabilities to other agents.We represent the liabilities in the form of a matrix, $L \in \mathbb{R}^{n\times n}$, where the uv^{th} entry l_{uv} represents the liability of u to v. We assume that $l_{uv} \geq 0,\ \forall u, v$ and $l_{uu} = 0,\ \forall u$. We denote the *total liabilities* of u by $L_u = \sum_{v\in N} l_{uv}$. Each agent u also has external assets (cash) of value $x_u \geq 0$.

The network can be viewed as a directed graph, where the nodes represent agents and the directed edges represent liabilities.

When the total liabilities of an agent exceed its incoming payments and external assets, we say that it is in *default*. Otherwise, we say that the agent is *solvent*. When an agent defaults, there may be losses associated with the process. This can include the payment of outstanding salaries, clearing fees, losses from inefficient asset liquidation, etc. These losses are often modeled using default cost parameters $\alpha, \beta \in [0, 1]$ [26]. A $1 - \alpha$ share of the agent's external assets and a $1 - \beta$ share of its incoming payments are lost from the network and only the remaining α and β shares can be used to make payments.

A tuple (N, L, x, α, β) is called a *financial network*. Note that $\alpha = \beta = 1$ corresponds to a model with no default costs. When an agent defaults, we assume (1) it pays out all of its external assets and incoming payments to its creditors, and (2) its payments are proportional to the size of its liabilities. So, the payment from u to v will be $p_{uv}(\boldsymbol{r}) = r_u l_{uv}$. Solvent agents pay all their liabilities in full. Given these payment principles, or *clearing rules*, the recovery rate vector and payments will satisfy the following definition.

Definition 1 (Clearing Recovery Rate Vector, CRR).

$$r_u = \min\left\{\frac{\bar{a}_u}{L_u}, 1\right\}$$

$$p_{uv} = r_u l_{uv}, \quad a_u = x_u + \sum_{v\in N} p_{vu}$$

$$\bar{a}_u = \begin{cases} a_u & \textit{if } a_u \geq L_u \\ \alpha x_u + \beta \sum_{v\in N} p_{vu} & \textit{if } a_u < L_u \end{cases}$$

Node u is solvent *if and only if* $r_u = 1 \Leftrightarrow a_u \geq L_u$.

Eisenberg and Noe [8] show that there is a unique CRR vector in financial networks without default costs. Rogers et al. [26] show that with default costs, the clearing vector may not be unique. They use a recursive algorithm to find the *greatest clearing vector*, which is the CRR vector such that no agent has a higher recovery rate in any other CRR vector[1]. From now on, we will assume that we always consider the greatest clearing vector if there is any ambiguity. This motivates the following definition.

[1] Since an increase in the recovery rate of one agent cannot decrease the recovery rates of others, the greatest clearing vector is well-defined and always exists. We refer to [26] for more details.

Definition 2 (Clearing Recovery Rate Vector Update Function). *Let (N, L, x, α, β) be a financial network. Define the update function $F : [0,1]^N \to [0,1]^N$, $\boldsymbol{r}^{k+1} = F(\boldsymbol{r}^k)$ with*

$$F_u(\boldsymbol{r}) := \begin{cases} 1 & \textit{if } a_u(\boldsymbol{r}) \geq L_u \\ \frac{\bar{a}_u(\boldsymbol{r})}{L_u} & \textit{if } a_u(\boldsymbol{r}) < L_u \end{cases}.$$

The recovery rate vector $\boldsymbol{r}$ is clearing if and only if it is a fixed point of the update function, that is, $F(\boldsymbol{r}) = \boldsymbol{r}$.

For our proofs, we need some basic properties of the update function F: The greatest clearing vector can be found by starting with $\boldsymbol{r}^0 = \mathbf{1}_N := (1, \ldots, 1)$ and applying F until convergence; F is bounded above by $\mathbf{1}_N$ and below by $\mathbf{0}_N$; and F is monotone, i.e., if $\tilde{\boldsymbol{r}} \leq \boldsymbol{r}$, then $F(\tilde{\boldsymbol{r}}) \leq F(\boldsymbol{r})$, where $\leq$ is to be understood elementwise. These properties follow directly from the results in [26], and we omit the proofs here. From this, by induction, we also have $F^n(\tilde{\boldsymbol{r}}) \leq F^n(\boldsymbol{r})$ in the limit as $n \to \infty$. In addition, for a financial network without default costs, we note that the total assets in the network are always preserved, as no money enters or leaves the network during clearing. We refer to this as the *conservation of total assets*.

Intuitively, the CRR vector is defined as the fixed point of the agents iteratively transferring the assets they have (and receive) to their creditors until they have paid their debts in full or have nothing left to give. Clearing allows us to evaluate the value of agents given the state of the network: It tells us what each agent would end up with if no changes were made to the network and all existing liabilities were *cleared*. Next, we define debt contracts and the lending game.

Definition 3 (Debt Contract). *A* debt contract *between l and b is a tuple (l, b, d, R), where l, the* lendor*, gives $d > 0$ external assets to b, the* borrower*, in return for a liability of $R \geq 0$. The contract results in:*

$$x'_l = x_l - d, \quad x'_b = x_b + d, \quad l'_{bl} = l_{bl} + R.$$

Definition 4 (Net Worth & Market Value). *The difference between the discounted assets and liabilities $(\bar{a}_u - L_u)$ of an agent is its* net worth*, denoted NW_u. We define the* market value *of an agent u as*

$$V_u = \bar{a}_u - r_u L_u = \max\left\{a_u - L_u, 0\right\} = \max\left\{NW_u, 0\right\}.$$

Definition 5 (Donation). *A donation from l to b is a debt contract with $R = 0$. l is* weakly (strongly) *incentivized to donate to b if this does not decrease (strictly increases) NW_l.*

Donations can also be seen as bailouts, sometimes referred to as bail-ins. Agents could have something to gain from saving other agents or making donations to other agents in the network, but we will not consider such cases in this paper.

Definition 6 (Break-Even Price). *We say that* $R_l^{BE}(b, d)$ *is a* break-even price *for a debt contract if* $NW_l' - NW_l = 0$.

Definition 7 (Lending Game). *Consider a financial network* (N, L, x, α, β) *and an agent* b *seeking to borrow* d *external assets. The* Lending Game *is defined as follows:*

Players:

- *Agents in the network* N *(excluding* b*).*

Actions:

- *Each agent* $u \in N \setminus \{b\}$ *can submit a bid* R_u *representing the liability it seeks from* b *in return for* d *external assets, or it can decide not to submit a bid.*

Objective:

- *Each agent aims to maximize its final net worth*

Rules:

- b *accepts the lowest bid if there is one.*
- *If multiple agents offer the lowest bid,* b *arbitrarily chooses the lender.*

Assumptions:

1. *There is* competition. *More precisely, there are at least two potential lenders, that is,* $|N \setminus \{b\}| \geq 2$, *and at least one potential lender has a break-even price less than* ∞.
2. *Donations are allowed, but there are no incentivized donations (remaining) in the original network.*
3. *Agents cannot count on receiving (incentivized) donations from others when calculating their net worth, but they can plan to make donations themselves.*

Nash Equilibrium (NE):

- $\{R_u^*\}_{u \in N \setminus \{b\}}$ *is a Nash Equilibrium (NE) if no agent* $u \in N \setminus \{b\}$, *can increase its net worth by unilaterally deviating from its current bid* R_u^*. *We call* R_u^* *an equilibrium price for agent* u.

Remarks:

- *The game models the lending process in the financial network, where agents compete to offer the lowest price for a debt contract with* b.
- *Agents must take into account the potential impact of the new debt contract when deciding on their bid price. Moreover, they have to compare winning the contract versus someone else winning the contract, rather than just comparing with the original network.*
- *The first assumption ensures that agents will be undercut if they offer more than another agent's break-even price.*

- *The second assumption ensures that no agent can benefit from giving an extremely favorable loan. Indeed, if a donation is incentivized, then there is no reason to wait for an opportunity to make a seemingly generous loan.*

Since changing non-minimum bids has no impact on the outcome, there can be infinitely many NEs. However, we are interested in the minimum price an agent would be willing to offer. For example, in a network with no liabilities, if b wants to borrow d and $x_b = 0$, then any price $R \geq d$ is a break-even price and all price vectors that are element-wise $\geq d$ are NEs. We are interested in the NE where all agents bid d exactly.

Observation 1 *(1) An agent can have infinitely many break-even prices for the same loan, and (2) There can be infinitely many Nash equilibria in a Lending Game.*

4 Results

This section presents results for lending in financial networks with and without default costs. We first prove two propositions that will be needed when proving the main results.

Proposition 1. *In a Lending Game where b wants to borrow $d > 0$, r_b is non-increasing in R for a given fixed lender $u \in N \setminus \{b\}$ (barring donations after the loan).*

Proof. This claim seems intuitive since increasing b's debts, whilst seemingly keeping its assets the same, cannot lead to an increase in b's recovery rate. However, note that u can gain from a higher R, and given a bankrupt u and a suitable topology, u's gain could translate into an overall gain for b.

Now we rule out such a scenario. Note that borrowing at a high price is a potentially inefficient way for b to transfer assets to u. b gets d more in external assets and R more in liabilities, a net loss of $R - d$; and u has d less in external assets and at most R more in incoming payments. If b is solvent after the loan, then the same asset transfer could be achieved by donating $R - d$ directly to u. And if b is bankrupt after the loan, then the transfer is inefficient since u gains less than b loses due to default costs. It would be better for b to donate $R - d$ directly. Note that we assume $\alpha \geq \beta$, so there is no advantage to having R as a liability instead of external assets. Therefore, since no incentivized donations exist, r_b is non-increasing in R. To be more formal, one would argue using the update function.

Proposition 2. *In a Lending Game, for all $u \in N \setminus \{b\}$, NW_u is non-decreasing in R when u wins the contract.*

Proof. We consider two regimes for R: (a) $R \leq a_b + d - L_b$, that is, b is solvent after the loan; and (b) $R > a_b + d - L_b$, i.e., b is in default after the loan.

Regime (a): b is solvent after the loan, so b repays all of its debts. If u makes the loan, then the incoming payments and therefore the total assets of u are increasing in R. Since neither the payments of b nor of u are decreasing in R, the recovery rates of all agents are non-decreasing in R. Therefore, also NW_u is non-decreasing in R.

Regime (b): From proposition 1, we know that r_b is non-increasing in R. However, note that when applying the update function F (definition 2), the total assets and therefore the total payments of b do not change as R is increased. However, the proportion of the payments going to u increases, while all the other creditors get proportionally less. The total payment gain for u is equal to the payment loss of all the other creditors combined; u is essentially "taking" the payments from the other creditors. Therefore, if u makes a loss due to some creditor of b not receiving enough payments, u could simply use (part of) the additional payments it has "taken" and donate them back to the creditors concerned. Therefore (with the appropriate donations) NW_u must be non-decreasing in R.

We have shown that NW_u is non-decreasing in R in the two regimes. To complete the proof, we have to consider the boundary of the two regimes. At the crossover point, we rely again on potential donations to show that there cannot be a drop in NW_u due to b going into default. Suppose the bankruptcy would lead to a drop in NW_u, then u could simply donate the difference ($R \leq a_b + d - L_b < d$) back to b and ensure that b does not go into default. This would lead to the same clearing vector as offering the lower price to start with, so there can be no drop in NW_u at the boundary.

We illustrate the two regimes and the different post-loan donation scenarios with an example in fig. 2. Note that these propositions are similar to the debt-swapping concept from [25] and Corollary 6 from [17].

4.1 No Default Costs

Theorem 1. *In a Lending Game with $\alpha = \beta = 1$ (no default costs), a break-even price for lending d to b always exists and the minimum break-even price is identical for all agents in the network. The minimum break-even price is:*

$$R^*(d) = \frac{d}{r_b}$$

where r_b is the recovery rate of b. A loan at this price does not affect the net worth of any agent other than b.

Proof. Let $u \in N \setminus b$ be an agent in the network and let $(N, L', x', 1, 1)$ be the financial network after the debt contract $(b, u, d, R = R^*(d))$ between b and u is agreed. We need to show the following: (a) The new loan does not affect the network (all recovery rates are unchanged); (b) $R^*(d)$ is a break-even price for u (and thus for all agents in $N \setminus b$); and (c) Any debt contract with $R < R^*(d)$ changes the net worth of u. To show (a), we show $\boldsymbol{r'} = \boldsymbol{r}$, which is also used to

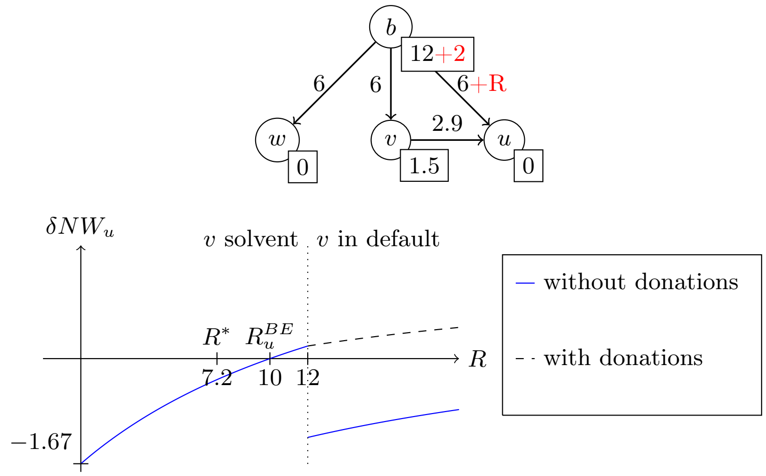

Fig. 2: An example illustrating how NW_u is increasing in R even when increasing R causes a detrimental bankruptcy. b borrows 2 from u in return for an additional liability of R. When R is too high, v defaults, affecting u. The default cost parameters are $\alpha = \beta = 0.5$. The blue line shows the net worth of u without post-loan donations, whereas the black dashed line takes donations into account — u bails out v. The y axis shows the *change* in NW_u relative to no loan. R^* indicates the NE price, R_u^{BE} the break-even price for u, and the dotted line the threshold above which v defaults.

show (b). From definition 3 we have that most of the network remains unchanged. To show that $\boldsymbol{r}' = \boldsymbol{r}$ we argue that $F'(\boldsymbol{r}) = \boldsymbol{r}$, where F' is the update function for the new network, which by definition 2 means that $\boldsymbol{r}$ is still a clearing vector of the new network. Since the clearing vector is unique, we get $\boldsymbol{r}' = \boldsymbol{r} \Leftrightarrow F'(\boldsymbol{r}) = \boldsymbol{r}$ [8, Theorem 2].

We evaluate $F'(\boldsymbol{r})$ element-wise for the elements $r_v, v \in N\backslash\{u, b\}$, r_u, and r_b. The assets and liabilities have not changed for all agents except x_u, x_b, l_{bu}, so $a_i'(\boldsymbol{r}) = a_i(\boldsymbol{r})$ and $L_i' = L_i$, and with definition 2 saying $a_v = a_v' \wedge L_v = L_v' \Rightarrow F_v'(\boldsymbol{r}) = r_v, \forall v \in N$ we have $r_v = r_v'$. Additionally, $L_u' = L_u$ and we have

$$a_u'(\boldsymbol{r}) = x_u' + \sum_{v \in N} r_v l_{vu}' x_u - d + \sum_{v \in N\backslash\{b\}} r_v l_{vu} + r_b \left(l_{bu} + \frac{d}{r_b} \right) = a_u(\boldsymbol{r}).$$

Thus, we have $r_u = r_u'$. The previous calculations could also be done for b, but it is simpler to use these results and consider r_b' according to definition 1:

$$r_b' = \min \left\{ \frac{x_b' + \sum_v r_v' l_{vb}'}{\sum_v l_{bv}'}, 1 \right\} = \min \left\{ \frac{r_b L_b + d}{L_b + \frac{d}{r_b}}, 1 \right\} = r_b.$$

Therefore, $\boldsymbol{r} = \boldsymbol{r}'$, showing that the network is not affected by the loan between u and b. For (b) we use from above that $a_u' = a_u$ and $L_u' = L_u$, so by definition 4 we get $NW_u' = NW_u$ showing that $R^*(d)$ is a break-even price for u.

It remains to prove (c). Suppose u offers the loan in exchange for a liability of $R < R^*(d)$, we show that the new net worth of u is then $NW''_u < NW'_u = NW_u$ (unless u is weakly incentivized to donate to b). Together with proposition 2, this gives that $R^*(d)$ is the minimum break-even price.

By proposition 2, we have $NW''_u \leq NW'_u = NW_u$, so assume $NW''_u = a''_u - L''_u = NW_u = a_u - L_u$. Since $L''_u = L_u$, we must have $a''_u = a_u$, which together implies that $r''_u = r_u$. We also have with proposition 1 that $1 \geq r''_b \geq r_b$, which when used in the update function F (definition 2) gives that $\boldsymbol{r}'' \geq \boldsymbol{r}' = \boldsymbol{r}$. With this assumption and the implications, we need to show that we get a contradiction or that u is weakly incentivized to donate to b. We now consider two cases: (1) b was solvent before initiating the loan, and (2) b was in default before initiating the loan.

Case (1): $r_b = 1 \Rightarrow r''_b = 1$, so $r''_b = r_b$ and thus $\boldsymbol{r}'' = \boldsymbol{r}$ which leads to a contradiction as:

$$NW''_u = a''_u - L''_u = a_u - d + R - L_u < a_u - L_u = NW_u.$$

Case (2): The update function F implies that if the recovery rates of b and u do not change with the addition of the debt contract, then the rest of the network remains unchanged. If $R = R^*(d) - \epsilon_1$ then u could instead donate $\epsilon_2 = \frac{(a_b+d)\epsilon_1}{L_b+R} = r''_b\epsilon_1 < \epsilon_1$ and make the loan $R^*(d)$ to b. Here, ϵ_2 is chosen s.t. the recovery rate of b is the same when lending at R and when lending at $R^*(d)$ with the donation of ϵ_2. But this implies that u is weakly incentivized to donate to b. Since the network is not changed by the loan, but only by the donation, then the donation was also weakly incentivized without the debt contract. This completes the proof of (c) and thus the theorem.

Theorem 1 can be extended to the following:

Theorem 2. *$R^*(d)$ is the equilibrium price for lending to b.*

That is, one can prove that in a financial network with competition, all agents offering to lend at $R^*(d)$ is an NE. Increasing the price would potentially benefit a lender, but then they would not win the contract, and decreasing the price would lower their net worth. We see that the network topology does not affect debt pricing in the basic model (beyond calculating the recovery rate vector). In particular, the lender's location does not matter. In the next section, we will see that this is also the case when we include default costs.

4.2 Default Costs

Theorem 1 shows that, without default costs, the minimum break-even price is the same for all agents in a network and that the break-even price is also an equilibrium price. However, with default costs, the break-even price is no longer the same for all potential lenders. Moreover, all agents offering their break-even price is not necessarily a Nash equilibrium.

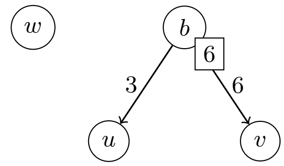

Fig. 3: A financial network where agents have different break-even prices. The default cost parameters are $\alpha = \beta = 1/2$. With $d = 2$, b would have 4 available after default costs have been deducted. Therefore, the break-even prices would be $R_u^{BE}(2) = 15$ for u and $R_w^{BE}(2) = 9$ for w. The break-even price for v would be infinite. Currently, v receives 2 after clearing and would need to receive 4 to break even, but b only has 4 available to pay out in total, so v would need to receive all of this.

Observation 2 *In a Lending Game with default costs, agents can have different break-even prices.*

This is illustrated in Figure 3. The break-even prices for lending 2 to agent b are 9, 15, and ∞ for w, u, and v, respectively. The intuition is that if b is in default, then some loaned value, more precisely $(1-\alpha)d$, is lost from the system, so a lender has to "take" the lost value from other creditors of b by increasing R. If an agent already has exposure to b, then it needs to raise the price further. We will see in Theorem 3 that, despite this, there is still a universal equilibrium price. This is because we also have to compare winning the contract with someone else winning the contract.

Observation 3 *In a Lending Game with default costs, agents offering their break-even price is not necessarily an NE.*

Consider the network in fig. 3 without agent w. Although agent v has no finite break-even price, it prefers to win the contract for $R = 9$ ($NW_v = 4/3$) than u winning the contract for $R = R_u^{BE} = 15$ ($NW_v = 1$). So bidding the break-even prices cannot be an NE.

As before, we assume that there are no (weakly) incentivized donations in the original network; however, it should be noted that with default costs, an agent could become strongly incentivized to donate to b. This is a surprising result; the network does not change, but just the prospect of a loan changes the donation incentives in the network.

Lemma 1. *There exist financial networks where v is not (weakly) incentivized to donate to b, but given the prospect of a loan from w to b, v becomes strongly incentivized to donate.*

While lemma 1 describes a valid scenario, we exclude it in the theorem, having already covered it with the lemma.

Definition 8 (Prospect of a loan). *An agent is* isolated *if they have no incoming or outgoing liabilities. The* prospect of a loan *is defined as the scenario* *in which a (new) isolated agent u offers a loan of d to agent b for a liability of $R = R_u^{BE}$.*

In addition, we have to restrict who can offer a loan and adapt what we mean by competition. The extreme cases of $\beta = 0, \alpha = 1$ and $\beta = 1, \alpha = 0$ illuminate the reasons for these changes. When $\beta = 1, \alpha = 0$, a bankrupt agent would offer the loan at an infinitesimal price since its external assets are otherwise lost anyway but the additional incoming liability will contribute to its (discounted) assets and therefore net worth. Therefore, we exclude bankrupt agents from offering loans. When $\beta = 0, \alpha = 1$, no agent in default would offer a loan, as the agent would gain nothing from the incoming liability, whilst losing d (discounted) assets. As a result, a single solvent agent would face no competition. Therefore, we redefine the notion of competition.

Definition 9 (Adapted Assumptions). *(A1) An agent can offer a loan if it remains solvent at the prospect of the loan. (A2) At least two agents can offer a loan, and at least one has a finite break-even price. (A3) There are no (weakly) incentivized donations at the prospect of a loan.*

Theorem 3. *In a Lending Game with the adapted assumptions (definition 9), there is an equilibrium price for all agents. Moreover, the marginal equilibrium price is*

$$\lim_{\epsilon \to 0} \left\{ \frac{R^*(\epsilon)}{\epsilon} \right\} = \frac{1}{r_b}.$$

That is, an agent would lend ϵ to b in return for a debt obligation of $\frac{\epsilon}{r_b}$, for ϵ small enough.

Proof. Let (N, L, x, α, β) be the financial network. If b is solvent, then u would lend d in return for a liability of d. The clearing vector does not change and u receives the payment in full in case of clearing. Since u is solvent (A1), NW_u is the same whether u has the d external assets or the d incoming payments. Therefore d is the equilibrium price. Therefore, we may assume that b is in default before the loan. To prove the theorem, we need to show: (a) Given some equilibrium price $R^*(d)$, all eligible agents offer $R^*(d)$. (b) the marginal price is $\frac{1}{r_b}$. If two agents u and v offer a loan, then a superscript ϕ denotes the winner. For example, $r_b^{\phi=u}$ is the recovery rate of b assuming u wins the contract. No superscript denotes the network before the new debt contract.

We first give an expression for $R^*(d)$. If u and v are offering the loan for $R^*(d)$ then the liabilities and external assets remain unchanged for all agents except b, u, and v. Furthermore, the liabilities of u and v are the same in both cases, and since $R^*(d)$ is assumed to be the equilibrium price, we have $r_u^{\phi=u} = r_u^{\phi=v}, \forall u \in N$ and $NW_u^{\phi=u} = NW_u^{\phi=v}, \forall u \in N$. Bank b also has the same assets and liabilities, and therefore also the same recovery rate in both cases: $r_b^{\phi=u} = r_b^{\phi=v}$. Since the liabilities of u are the same, we only need to consider u's assets when calculating its new net worth. We have:

$$\begin{aligned} a_u^{\phi=v} &= x_u + \sum_w r_w^{\phi=v} l_{wu} = a_u^{\phi=u} \\ &= x_u - d + \sum_w r_u^{\phi=u} l_{wu} + r_b^{\phi=u} R^*(d) \Leftrightarrow R^*(d) = \frac{d}{r_b^{\phi=u}}. \end{aligned}$$

Note that $r_b^{\phi=u} = r_b^{\phi=v}$, so $R^*(d)$ is independent of ϕ. Finally, by (A3), we do not have to consider donations.

To show (a) we need to show that no eligible agent would deviate from offering $R^*(d)$. We consider an eligible agent u. If u increases its price, then its offer will not be accepted so NW_u will not increase. On the other hand, let us consider u decreasing its offer to $\widehat{R_u} < R^*(d)$. We have shown above that $NW_u^{\phi=u} = NW_u^{\phi=v}$, when $R_u = R_v = R^*(d)$. Moreover, by proposition 2, we have $\widehat{NW}_u^{\phi=u} \leq NW_u^{\phi=u}$. Combining these, we find that even though reducing the price guarantees that u wins the contract, its net worth cannot be greater: $\widehat{NW}_u^{\phi=u} \leq NW_u^{\phi=v}$.

It remains to show (b), i.e., the marginal price as $d \to 0$. If $r_b = 1$, then $R^*(d) = d$ and $r_b^{\phi=u} = 1 = r_b$. Otherwise,

$$r_b^{\phi=u} = \frac{\bar{a}_b^{\phi=u}}{\sum_i l_{bi}^{\phi=u}} = \frac{r_b L_b + \alpha d}{L_b + \frac{d}{r_b^{\phi=u}}} = \frac{r_b L_b - (1-\alpha) d}{L_b}.$$

Since $r_b < 1$, $L_b > 0$. Therefore, taking the limit $d \to 0$ gives $r_b^{\phi=u} = r_b$. This gives the limit equilibrium price as required: $\lim_{\epsilon \to 0} \frac{R^*(\epsilon)}{\epsilon} = \frac{1}{r_b}$.

Note that $R^*(d) = \frac{L_b d}{r_b L_b - (1-\alpha) d}$. So, as $d \to \frac{r_b L_b}{1-\alpha}$, the equilibrium price goes to infinity. Thus, no agent would ever lend more than $d = \frac{r_b L_b}{1-\alpha}$ to b at any price.

Observation 4 *$R^*(d)$ is the break-even price for an* isolated *agent lending d to agent b.*

In other words, this is the price that an agent would charge if there were no network effects to consider. This follows from the calculations in the proof.

Calculating the impact of a debt contract on the value of an agent can be complicated, but the proof relies on the fact that comparing the value of an agent when it wins the contract with the value when somebody else wins the contract (for the same price) is relatively easy.

Observation 5 *We can omit (A1) by restricting to $\alpha = \beta$. This eliminates the extreme cases mentioned earlier.*

The proof in part (a) can be adapted as follows. Take an agent u (may be in default), where there is no path of liabilities from v to u. This case is the same as in the proof, as losses incurred by v cannot impact u or its debtors. The other case is that there is a path from v to u, which is split into the cases: (i) no path from u to v, (ii) path from u to v. For (i), u can donate its profits to v, but v cannot expect this according to the definition of a Lending Game. So u can match v's offer, but v would undercut u. For (ii), u can donate to v. Assuming that u does not have to donate everything, then it can undercut the price offered by v. Otherwise, u does not care if it wins and donates everything or v wins. From v's perspective, its net worth will be lower if u wins but does not donate. So v would rather undercut u and ensure that it wins the contract.

4.3 Stability of the Network

The addition of default costs means that some capital will disappear from the system, which can have negative effects on an unstable network. Showcased by the following two theorems with proofs the appendix.

Theorem 4. *For any $d > 0$, $p < 1$ and $\alpha, \beta < 1$ there exist financial networks of size n with default costs (α, β) where a loan from u to b of amount d will bankrupt pn of the agents, including u (barring donations).*

Observation 6 *In the proof of theorem 4, v becomes strongly incentivized to donate to i_1 at the prospect of a loan.*

Observation 7 *As an extension to theorem 4, when donations up to the net worth of agents are allowed, the example can be adapted so that no agent is both incentivized and able to bail out i_1.*

Observation 8 *Lending does not depend on network effects; however, it is strongly tied to the donation game, which does depend on network effects. As such, if an agent can only observe the network in a finite radius of itself and b, then it cannot protect itself from the effect seen in theorem 4.*

We now show that even networks with a minimum liquidity requirement can be unstable. For the next theorem, we need the following definition.

Definition 10 (Minimum liquidity). *A network is said to have γ-minimum liquidity for $\gamma \in (0, 1)$, if $x_i \geq \gamma L_i, \forall i \in N$.*

Theorem 5. *For any $d > 0$, $p < 1$, $0 < \gamma < 1$, and $\alpha, \beta < 1$ there exists a financial network of size n with default costs (α, β) and minimum liquidity γ, where a loan from u to b of amount d will bankrupt pn of the agents, incl. u.*

This section has shown how the addition of a new loan in a financial network with default costs can cause a net negative for the system. Two examples were constructed to show that the loan can cause arbitrarily large losses in any financial system with default costs even when forcing agents to keep a fraction of their outgoing liabilities in cash.

5 Conclusion

In this paper, we ask what the right price is for lending in financial networks. There are various models of financial networks, and we consider lending in the two most established models from the literature. Given a fixed clearing mechanism, we base the price of a debt contract on its effect on the value (upon clearing) of the lender. Moreover, we consider competing agents, since contracts between other agents can also impact the net worth.

In the first model, we show that every agent in the network has the same break-even price and as such the same equilibrium price for lending to a specific agent. When we add default costs to the network, we find that agents may now have different break-even prices, but surprisingly, they would all offer the same price at equilibrium. Moreover, the equilibrium price is the same as the break-even price of a (hypothetical) isolated agent.

Finally, we show that a financial network with default costs can be very fragile to lending in the sense that a new loan can cause a cascade of defaults. Requiring agents to maintain a fixed proportion of their liabilities as external assets does not fix this problem, calling into question the efficacy of international banking standards.

There are many potential directions for future research. One could extend this work to multiple concurrent borrowers. Similarly, one could consider introducing cross-ownership, stochastic assets, or other financial instruments, which may have interesting effects on debt pricing.

References

1. Allen, F., Gale, D.: Financial contagion. Journal of Political Economy **108**(1), 1–33 (2000), http://www.jstor.org/stable/10.1086/262109
2. Anand, K., Bédard-Pagé, G., Traclet, V.: Stress testing the canadian banking system : a system-wide approach (2014)
3. Banerjee, T., Feinstein, Z.: Pricing of debt and equity in a financial network with comonotonic endowments. arXiv preprint arXiv:1810.01372 (2018)
4. Battiston, S., Puliga, M., Kaushik, R., Tasca, P., Caldarelli, G.: Debtrank: Too central to fail? financial networks, the fed and systemic risk. Scientific reports **2**, 541 (2012)
5. Bernard, B., Capponi, A., Stiglitz, J.E.: Bail-ins and bail-outs: Incentives, connectivity, and systemic stability. Working Paper 23747, National Bureau of Economic Research (8 2017). https://doi.org/10.3386/w23747
6. Bluhm, C., Overbeck, L., Wagner, C.: Introduction to credit risk modeling. Crc Press (2016)
7. Cifuentes, R., Ferrucci, G., Shin, H.S.: Liquidity risk and contagion. Journal of the European Economic Association **3**(2/3), 556–566 (2005)
8. Eisenberg, L., Noe, T.H.: Systemic risk in financial systems. Management Science **47**(2), 236–249 (2001), http://www.jstor.org/stable/2661572
9. Elsinger, H., Lehar, A., Summer, M.: Network models and systemic risk assessment. See Fouque&Langsam pp. 287–305 (01 2013)

10. Fischer, T.: No-arbitrage pricing under systemic risk: Accounting for cross-ownership. Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics **24**(1), 97–124 (2014)
11. Freixas, X., Parigi, B.M., Rochet, J.C.: Systemic risk, interbank relations, and liquidity provision by the central bank. Journal of Money, Credit and Banking **32**(3), 611–638 (2000), http://www.jstor.org/stable/2601198
12. Friedetzky, T., Kutner, D.C., Mertzios, G.B., Stewart, I.A., Trehan, A.: Payment scheduling in the interval debt model. In: International Conference on Current Trends in Theory and Practice of Computer Science. pp. 267–282. Springer (2023)
13. Gai, P., Haldane, A., Kapadia, S.: Complexity, concentration and contagion. Journal of Monetary Economics **58**(5), 453–470 (2011)
14. Gai, P., Kapadia, S.: Contagion in financial networks. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **466**(2120) (2010)
15. Glasserman, P., Young, H.P.: How likely is contagion in financial networks? Journal of Banking & Finance **50**(C), 383–399 (2015)
16. Glasserman, P., Young, H.P.: Contagion in financial networks. Journal of Economic Literature **54**(3), 779–831 (2016)
17. Hoefer, M., Ventre, C., Wilhelmi, L.: Algorithms for Claims Trading. In: 41st International Symposium on Theoretical Aspects of Computer Science (STACS) (2024)
18. Jackson, M.O., Pernoud, A.: Systemic risk in financial networks: A survey. Annual Review of Economics **13**, 171–202 (2021)
19. Kanellopoulos, P., Kyropoulou, M., Zhou, H.: Debt transfers in financial networks: Complexity and equilibria. In: Proceedings of the 2023 International Conference on Autonomous Agents and Multiagent Systems. pp. 260–268 (2023)
20. Lando, D.: Credit risk modeling. In: Handbook of Financial Time Series, pp. 787–798. Springer (2009)
21. McNeil, A.J., Frey, R., Embrechts, P.: Quantitative risk management: concepts, techniques and tools-revised edition. Princeton university press (2015)
22. Merton, R.C.: On the pricing of corporate debt: The risk structure of interest rates. The Journal of finance **29**(2), 449–470 (1974)
23. Papachristou, M., Kleinberg, J.: Allocating stimulus checks in times of crisis. In: Proceedings of the ACM Web Conference 2022. pp. 16–26 (2022)
24. Papp, P.A., Wattenhofer, R.: Default ambiguity: Finding the best solution to the clearing problem (2020)
25. Papp, P.A., Wattenhofer, R.: Debt Swapping for Risk Mitigation in Financial Networks. In: 22nd ACM Conference on Economics and Computation (EC), Budapest, Hungary (07 2021)
26. Rogers, L.C.G., Veraart, L.A.M.: Failure and rescue in an interbank network. Management Science **59**(4), 882–898 (2013)
27. Roukny, T., Bersini, H., Pirotte, H., Caldarelli, G., Battiston, S.: Default cascades in complex networks: Topology and systemic risk. Scientific reports **3**(1) (2013)
28. Schuldenzucker, S., Seuken, S., Battiston, S.: Clearing payments in financial networks with credit default swaps [extended abstract]. In: Proceedings of the 2016 ACM Conference on Economics and Computation. p. 759. EC '16, Association for Computing Machinery (2016). https://doi.org/10.1145/2940716.2940791
29. Siebenbrunner, C., Sigmund, M.: Do interbank markets price systemic risk? (10 2018). https://doi.org/10.13140/RG.2.2.26958.31048
30. Suzuki, T.: Valuing corporate debt: the effect of cross-holdings of stock and debt. Journal of the Operations Research Society of Japan **45**(2), 123–144 (2002)
31. Upper, C.: Simulation methods to assess the danger of contagion in interbank markets. Journal of Financial Stability **7**, 111–125 (08 2011)

6 Appendix

Proof of lemma 1

Proof. Consider the network in Figure 4. Ignoring external assets, $NW_v = 2$ initially. If v donates 9 to b, then NW_v drops to 1. Thus, v is not incentivized to bail out b. However, if w offers a loan of $d = 4$ for a debt of $R = R_w^{BE} = 60$, then NW_v drops to 2/3.

Proof of Theorem 4

Proof. To prove this, we construct such a network. Without loss of generality, the expressions can be simplified by setting $\alpha = \beta = \max(\alpha, \beta)$. A sketch of the network can be seen in Figure 5, and before the new debt contract is formed, only bank b is in default with $r_b = \frac{\alpha}{C}$ while the nodes $i_2, i_3, ..., i_K$ merely pass along their incoming payments while remaining solvent.

After forming the debt contract, the recovery rate for bank b changes to $\hat{r}_b = \frac{\alpha(1+d)}{C(1+\frac{d}{\alpha})} < \frac{\alpha}{C}$. Ignoring the network effects for a moment, the change in the net worth of bank u when providing the loan is

$$\Delta V_u = -d + \hat{r}_b \frac{Cd}{\alpha} = -d + \frac{\alpha(1+d)}{C(1+\frac{d}{\alpha})} \frac{Cd}{\alpha} = -d + \frac{1+d}{1+\frac{d}{\alpha}} d < 0.$$

Therefore, the debt contract presents a loss for bank u when ignoring the effect of the contract on other banks. Now consider the assets, liabilities, and recovery rate of bank i_1 after the debt contract:

$$\hat{a}_{i_1} = C - \alpha + \hat{r}_b C = C - \alpha + \frac{\alpha(1+d)}{C(1+\frac{d}{\alpha})} C < C - \alpha + \alpha = C = L_{i_1} = \hat{L}_{i_1}$$

$$\hat{r}_{i_1} = \frac{\alpha(C - \alpha + \hat{r}_b C)}{C} = \frac{\alpha\left(C - \alpha + \frac{\alpha(1+d)}{1+\frac{d}{\alpha}}\right)}{C} < \alpha.$$

Due to the layout of the network, if bank i_k is in default with recovery rate $\hat{r}_{i_k}$ then bank i_{k+1} is also in default. This can be quickly seen as $\hat{r}_{i_k} < 1$ means i_{k+1}

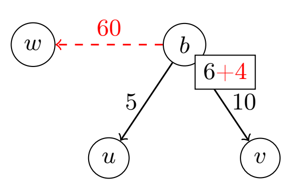

Fig. 4: An example financial network illustrating lemma 1. The default cost parameters are $\alpha = \beta = 1/2$. The net worth of v is initially 2. If b wants a loan of 4 from w then the break-even price for w would be $R_w^{BE}(4) = 60$. Adding this loan would drop the net worth of v from 2 to 2/3, while a donation of 9 from v to b would result in a net worth of 1 for v. Therefore, v is strongly incentivized to donate when the potential loan is taken into consideration.

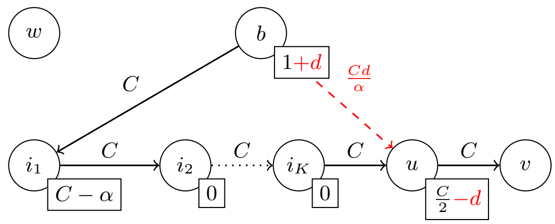

Fig. 5: Network satisfying theorem 4. $C > 1$ and $K \geq \left\lceil \frac{-\log 3}{\log \alpha} \right\rceil$. The dashed, red edge marks the newly formed debt contract, which causes the system to collapse. The dotted lines indicate paths of identical agents, with 0 external assets and liabilities of C to the next agent along the path. The agent w ensures there is competition. The contract could also be to w with the same effect.

receives less than C and therefore cannot fulfill its own obligation of C. The recovery rate for bank i_{k+1} is then given by

$$\hat{r}_{i_{k+1}} = \frac{\alpha C \hat{r}_{i_k}}{C} = \alpha \hat{r}_{i_k}.$$

Using the above and that i_1 is in default, we get

$$\hat{r}_{i_K} = \alpha^{K-1} \hat{r}_{i_1} < \alpha^K.$$

Setting $K \geq \left\lceil \frac{-\log 3}{\log \alpha} \right\rceil$, it follows that $\hat{r}_{i_K} < \alpha^K \leq \frac{1}{3}$. This, in turn, implies that the assets for u fall below C, causing u to also default.

In the network, there are $K+3$ banks, 1 was initially in default, and $K+2$ are now in default, so $K+1$ banks went into default due to the loan. We need $\frac{K+1}{K+3} \geq p \Rightarrow K \geq \frac{3p-1}{1-p}$.

Proof of Theorem 5

Proof. The network in Figure 5 will show this with some minor modifications. The liabilities of $i_k, \forall k = 1, ..., K$ are given by $C_k = \alpha \frac{1}{(1-\gamma)^k} C_0$, with C_0 being the liability from b to i_1. The liability from u to v is $C = \frac{d+B+C_K}{1-\gamma}$. In addition, the assets are $a_b = \gamma C_0$, $a_k = \gamma C_k$, and $a_g = \gamma C + d + B, B \geq 0$. After forming the debt contract $i_1, i_2, ..., i_K, u$ all defaults.