The Impact of Core Constraints on Truthful Bidding in Combinatorial Auctions

Robin Fritsch[a], Younjoo Lee[b], Adrian Meier[a], Kanye Ye Wang[c,*], Roger Wattenhofer[a]

[a] *ETH Zurich, Zurich, Switzerland*
[b] *Seoul National University, Seoul, Korea*
[c] *University of Macau, Macau, China*

Abstract

Combinatorial auctions (CAs) offer the flexibility for bidders to articulate complex preferences when competing for multiple assets. However, the behavior of bidders under different payment rules is often unclear. Our research explores the relationship between core constraints and several core-selecting payment rules. Specifically, we examine the natural and desirable property of payment rules of being *non-decreasing*, which ensures that bidding higher does not lead to lower payments. Earlier studies revealed that the VCG-nearest payment method – a commonly employed payment rule – fails to adhere to this principle even for single-minded CAs. We establish that when a *single effective core constraint* exists, the payment maintains the non-decreasing property in single-minded CAs. To identify auctions where such a constraint is present, we introduce a novel framework using conflict graphs to represent single-minded CAs and establish sufficient conditions for the existence of single effective core constraints. We proceed with an analysis of the implications on bidder behavior, demonstrating that there is no overbidding in any Nash equilibrium when considering non-decreasing core-selecting payment rules. Our study concludes by establishing the non-decreasing nature of two additional payment rules, namely the proxy and proportional payment rules, for single-minded CAs.

Keywords: Combinatorial auctions, core-selecting payment rules, non-decreasing payment rules

---

1. Introduction

Combinatorial auctions (CAs) serve as a vital tool in allocating multiple, individually-valued assets at market-driven prices, as established in foundational literature [1]. These auctions allow bidders to place bids on sets, or *bundles* of items, as opposed to restricting them to single-item bids. This enables participants to express complex preferences for multiple items including interactions between them.

The auction mechanism aims to find the most efficient allocation of items to bidders while maximizing overall value. The mechanism of a CA is built upon two primary components: an algorithm responsible for designating the winning bids and a payment scheme that defines the financial obligations of the winning bidders.

CAs have seen widespread adoption, with applications ranging from spectrum allocation to energy markets, and transactions at times exceeding billions of dollars in

*Corresponding author.

Email addresses: rfritsch@ethz.ch (Robin Fritsch), youlee@student.ethz.ch (Younjoo Lee), meiera@student.ethz.ch (Adrian Meier), wangye@um.edu.mo (Kanye Ye Wang), wattenhofer@ethz.ch (Roger Wattenhofer)

volume [2, 3]. A critical objective auction designers is to foster truthfulness, ensuring that bidders are motivated to disclose their genuine valuation of the assets in question. The Vickrey-Clarke-Groves (VCG) scheme [4, 5, 6] stands out as the only mechanism capable of ensuring truthful bidding when optimal welfare allocation is in place. However, the real-world applicability of VCG is often limited due to its tendency to result in payments that are unreasonably low [7].

	Local bidder 1	Local bidder 2	Global bidder
Bundle	$\{A\}$	$\{B\}$	$\{A, B\}$
Bid	6	7	9
Allocation	$\{A\}$	$\{B\}$	$\{\}$
First-price payment	6	7	0
VCG payment	2	3	0
VN payment	4	5	0

Table 1: An example of an auction with 3 bidders and 2 items. The two local bidders win the auction because they bid $6 + 7 = 13$, whereas the global bidder only bids 9.

To illustrate, refer to Table 1, which showcases an example of a combinatorial auction, namely a Local-Local-Global (LLG) auction setup. In this scenario, two bidders have localized interests, each targeting just a single item, while a third "global" bidder seeks to acquire all items. Since the two local bidders combined bid more than the global bidder, the local bidders win the auction. Utilizing a first-price payment mechanism in this instance would result in the local bidders collectively paying $6 + 7 = 13$. Such an outcome makes it advantageous for them to misrepresent their true intentions, perhaps by jointly bidding merely 10.

Furthermore, the example illustrates how comparatively low VCG payments would be. In fact, in our example scenario, the VCG payments of $2+3 = 5$ fall short of the value put forth by the global bidder. This mismatch suggests that the global bidder and the seller might prefer to sidestep the VCG mechanism and negotiate a direct transaction.

In response to these challenges, core-selecting payment rules have emerged, notably the VCG-nearest (VN) payment rule [8] which is frequently used in spectrum auctions [3]. These approaches aim to ensure that the auctioneer secures a fair return on the auctioned items [9]. Specifically, the VN rule identifies the point closest to the VCG payments within the *core*. Here, the *core* refers to the spectrum of payment amounts where no group of bidders is willing to exceed what the winners are paying [10] (refer to Figure 1).

Nonetheless, even core-selecting payment methods like the VN payment are not without flaws. Previous research has demonstrated instances where bidders could reduce their payments by submitting inflated bids above their true valuation under the VN payment rule [11]. We refer to such behavior as *overbidding.*

In this paper, we study the mechanics of payment rules within the scope of single-minded CAs, where each participant targets a single known set of items. While overbidding occurs in the single-minded setting, such occurrences are so far only known for complex arrangements of multiple goods and participants [11]. We set out to assess this complexity threshold of single-minded CAs: At what point of auction complexity do VN payments cease to incentivize bidders to make artificially higher bids? This question takes on increased importance in light of evidence that bidders can sometimes benefit from overbidding.

Additionally, we investigate the relationship between core constraints and core-selecting payments mechanisms. Additionally, we delve into the intricate relationship

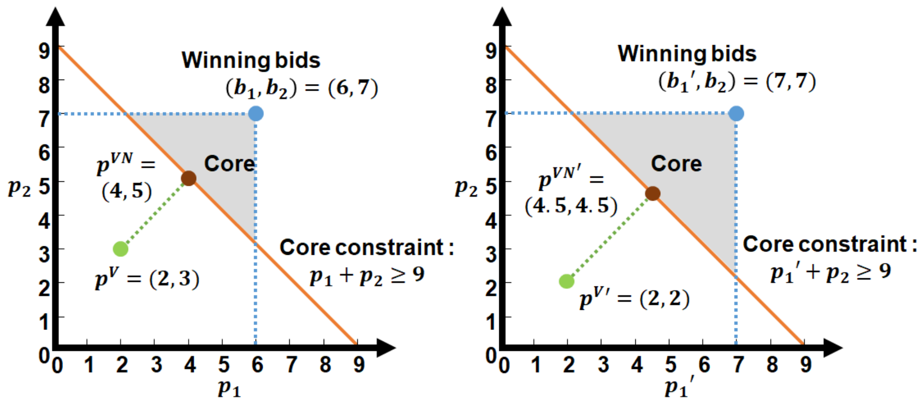

Figure 1: Left: payment space of winning bidders in Table 1. The green point p^V is the VCG payment point, the red point P^{VN} is the VN payment point, the orange line is the core constraint on payments of local bidders 1 and 2, and the gray triangle is the core given by core constraints. Right: If bidder 1 increases their bid from 6 to 7, their payment increases as well, from 4 to 4.5.

between core constraints – those linear limitations that define the core – and core-selecting payment mechanisms. The core constraints are a set of linear constraints generated by the bidders' bids on their desired bundles, which collectively form a geometric polytope (the *core*). By doing so, we aim to offer a nuanced understanding of how these elements interact, and whether specific types of core constraints are more conducive to the effective functioning of core-selecting payment rules. Our results can be summarized as follows.

Firstly, we examine the non-decreasing property, a highly sought-after characteristic in the context of payment rules. This attribute stipulates that elevating a bid should not result in a lowered payment obligation for the bidder. Our exploration focuses on identifying the types of core constraints under which VN payments adhere to this non-decreasing principle. We establish that whenever a single effective core constraint is present, the VN payments are inherently non-decreasing.

Secondly, we make progress in pinpointing which auction configurations inherently follow the non-decreasing property. To facilitate this, we apply a graph-centric model to represent the complexities of CAs. Our model – termed the *conflict graph* – illuminates overlaps in the bidders' preferred bundles. We identify certain conditions in these conflict graphs that guarantee the presence of a single effective core constraint. Specifically, our findings reveal that a complete multipartite graph or a maximal independent set with no more than two nodes in the conflict graph ensures this criterion. Moreover, in auctions with a maximum of three winners, VN payments maintain their non-decreasing nature, irrespective of the presence of a single effective core constraint. While we do not provide a proof, we conjecture this fact to be true for up to five winners. This would imply that the known example of overbidding with 6 winners [11] is minimal in terms of the number of winners.

Thirdly, we examine how these payment rules influence bidders' strategic choices. Our analysis confirms that in single-minded CAs, a non-decreasing payment rule inherently discourages overbidding strategies, rendering them weakly dominated by truthful bidding in any Nash equilibrium. This establishes the validity of a conjecture previously put forth by Bosshard et al. [11].

Lastly, our inquiry extends to evaluating the non-decreasing attributes of two other prevalent payment rules: the proxy payment and the proportional payment. Though these payment rules fail to maintain the non-decreasing property in multi-minded bidding scenarios, we demonstrate that they do satisfy this condition when applied to single-minded CAs.

In summary, this paper adds several key contributions to the ongoing research surrounding CAs. These include the proof that VN payments maintain the non-decreasing property under specific core constraints, identification of conditions within conflict graphs that dictate the presence of a single effective core constraint, and evidence dispelling the viability of overbidding strategies under non-decreasing payment rules in any Nash equilibrium.

2. Related Work

The complexities tied to the motivations of participants in combinatorial auctions featuring core-selecting payment rules are not yet fully understood [12]. Earlier works, such as those by Day and Milgrom [10], argue that core-selecting payment minimize incentives to false report. However, particular conditions under which specific incentives, such as the non-decreasing trait, prevail remain unclear.

In earlier analyses, the VN payment rule has been noted for its non-decreasing behavior, specifically in LLG auctions [13]. However, the property is not universal across all types of single-minded CAs [11]. Investigations by Markakis and Tsikirdis on alternative payment rules – specifically, 0-nearest and b-nearest – also show that these rules maintain a non-decreasing characteristic in single-minded CAs [14]. Our research augments this understanding by identifying conditions for which the prevalent VCG-nearest payment rule maintains non-decreasing behavior in these auctions.

Regarding the computational aspects of core-selecting payment rules, Day and Raghavan propose a constraint generation to express the pricing problem concisely [9]. A faster algorithmic approach to core constraint generation, based on the conflict graph among auction participants, was later proposed by Bünz et al. [15].

The influence of payment attributes on participants' strategic maneuvers in CAs cannot be overstated. Earlier work has explored game-theoretical analyses [9] and has highlighted both under-bidding and over-bidding strategies. Sano's research indicates that sincerity is not the dominant strategy in particular types of core-selecting auctions [16], while the question of whether over-bidding strategies consistently appear in Nash equilibria remains open.

Existing literature has shown that under-bidding strategies are invariably present in both pure Nash equilibria (PNE) and Bayesian Nash equilibria (BNE) in core-selecting CAs. Further studies by Beck and Ott on full-information settings confirmed that each minimum-revenue core-selecting CA contains a PNE that consists only of over-bids [17]. Yet, the motivations for over-bidding in the context of privately held valuations are not adequately explored.

3. Mathematical Framework

We examine auction settings wherein all participating bidders and the auctioneer operate independently, and guided by rational and self-centered objectives. In particular, bidder focus on maximizing their individual utility.

3.1. Structure of Combinatorial Auctions

This work deals with combinatorial auctions (CAs), where goods in a set $M = \{1, \ldots, m\}$ are allocated among bidders represented by a set $N = \{1, \ldots, n\}$. Specifically, we are concerned with single-minded combinatorial auctions (SMCAs), where each bidder targets a unique bundle of goods. The bundle that the i^{th} bidder is targeting is denoted by $k_i \subset M$, and the collection of all such bundles is given by the tuple $k = (k_1, \ldots, k_n)$. This tuple, known as the auction's interest profile, is assumed to be both known and unchanging. In addition, each bidder i has a private valuation $v_i \in \mathbb{R}\geq 0$ of the bundle they are bidding for, and submits a corresponding bid $b_i \in \mathbb{R}\geq 0$. The collective bids constitute the bid profile $b = (b_1, \ldots, b_n)$, with b_{-i} representing all bids other than bidder i's. Moreover, b_L denotes the bids of a subset $L \subset M$ of bidders.

A CA mechanism, denoted (X, P), is composed of an algorithm X for selecting winners and a payment scheme P that sets the amount due by each winning bidder.

3.2. Winner Selection

The allocation function $X(b)$ yields an optimal set x that contains bidders receiving their preferred bundles, while the rest end up empty-handed. The efficiency criterion is that the allocation maximizes the sum of all successful bids, formally defined as the reported social welfare $W(b, x) = \sum_{i \in x} b_i$. This optimization is subject to the constraint that each good is part of at most one winning bundle.

In this setup, bidders aim to optimize their utility, defined as the value they assign to the acquired bundle minus the associated payment. Due to the private nature of the valuations, the auctioneer aims to maximize reported social welfare rather than actual social utility.

3.3. Types of Payment Rules

We posit that the payment function adheres to a voluntary participation constraint; that is, no bidder will be charged more than their bid, formalized as $p_i \leq b_i$ for every $i \in N$.

Remember that in CAs, Vickrey-Clarke-Groves (VCG) payments ensure truthful bidding, i.e. bidding their true valuation is a weakly dominant strategy for bidders. For an efficient allocation $x = X(b)$, the VCG payment for bidder i, denoted p_i^V, is given by:

$$p_i^V(b, x) = W(b, X(b_{-i})) - W(b, x_{-i})$$

Here, x_{-i} represents all winners except bidder i and $X(b_{-i})$ is the set of winners in the auction excluding bidder i.

All payment rules we consider are *core-selecting*, i.e. return a payment within the *core*. The core is demarcated by lower-bound constraints (core restrictions), preventing any coalition from mutually beneficial renegotiation.

Definition 3.1 (Core-selecting payment method)**.** For an optimal set $x = X(b)$, the core comprises all payment vectors $p(b, x)$ that satisfy the subsequent core constraints for any subset $L \subseteq N$:

$$\sum_{i \in N \setminus L} p_i(b, x) \geq W(b, X(b_L)) - W(b, x_L)$$

with $x_L = x \cap L$ being the set of winning bidders in L under the allocation x, and $X(b_L)$ being the set of winners in the auction with only the bids of bidders in L.

A payment rule is called core-selecting if it selects a point within the core. Lastly, the minimum-revenue core, a subset of the core, contains all points with the smallest total payment among points within the core.

In particular, VN payments locate a point in the minimum-revenue core nearest to the VCG point, following the definition in previous work [11, 14] [1].

[1]Other work [8] does not explicitly require the VN payment to be in the minimum-revenue core, but instead only mandates it to be part of the core. Note here that for simple auctions, such as LLG, the VN payments already necessarily falls into the minimum-revenue core without needing to explicitly specify this.

Definition 3.2 (VCG-nearest payment)**.** The VCG-nearest payment rule (VN payment, quadratic payment, closest-to-VCG payment) selects the point within the minimum-revenue core closest to the VCG point based on Euclidean distance.

We also investigate the equal-share proxy payment and proportional payment, both of which are core-compliant.

Definition 3.3 (Equal-share proxy payment)**.** The equal-share proxy payment selects a core-compliant point where winners divide the total payment roughly equally. The payment can be represented as $p_i(b, x) = \min[\alpha, b_i]$ for the smallest $\alpha \geq 0$ satisfying the core constraints.

Definition 3.4 (Proportional payment)**.** The proportional payment assigns payments proportional to the winning bids within the core. The payment is of the form $p_i(b, x) = \alpha \cdot b_i$ with $\alpha \in [0, 1]$ for the smallest $\alpha \geq 0$ satisfying the core constraints.

Finally, recall the definition of the non-decreasing property of a payment rule.

Definition 3.5 (Non-decreasing payment rule)**.** For an allocation x, we write $\mathcal{B}_x$ for the set of bid profiles for which x is efficient. A payment function p is non-decreasing if, for any given allocation x and for any bidder i, for any pair of bid profiles $b, b' \in \mathcal{B}_x$ such that $b'_i \geq b_i$ and $b_{-i} = b'_{-i}$, we have:

$$p_i(b', x) \geq p_i(b, x)$$

4. Single Effective Core Constraints

We begin by establishing a sufficient condition on core constraints that ensures that the VN payment rule consistently exhibits non-decreasing behavior.

Core-selecting payment rules operate within boundaries defined by core constraints, which set a lower limit on payments to prevent any coalition from achieving a higher reported welfare than the actual winners. However, not all constraints exert an equal impact – some become irrelevant as they are automatically satisfied by other, more stringent, constraints. For instance, in an LLG auction as depicted in Figure 1, where local bidders emerge victorious, we have:

$$p_1 + p_2 \geq b_G \tag{1}$$
$$p_1 \geq b_G - b_2 \tag{2}$$
$$p_2 \geq b_G - b_1 \tag{3}$$

Here, b_G represents the global bidder's bid. Constraints (2) and (3) are effectively subsumed by (1), thereby rendering them redundant. For constraint (2), this follows from $p_2 \leq b_2$, and equivalently for (3). This leads us to conceptualize the notion of a single effective core constraint, which we formally define below.

We also note that core constraints of the form $p_i \geq p_i^V$ – which we term VCG-constraints – such as constraints (2) and (3) can be neglected when computing VN payments. Minimizing the distance from p^V to p already averts $p_i < p_i^V$.

Definition 4.1. In the context of an SMCA with a predetermined interest profiles and a fixed winner allocations, a single effective core constraint (SECC) is said to be present if satisfying one specific core constraint automatically fulfills all others. Formally, an SECC exists if, for all bid profiles, the polygon shaped by this unique core constraint and the voluntary participation constraints exactly matches the core, as specified by the full set of core constraints.

Theorem 4.2. *The VN payment rule is non-decreasing for SMCAs with a single effective core constraint.*

Proof. To prove this theorem we will first compute an explicit formula for the VN payments. The payments of all losing bidders are 0. For all winning bidders whose payment is not bound from below by the SECC, the VN payment simply equals the VCG payment. Let S be the set of winners whose payment is bound from below by the SECC. Then we have the following constraints on the VN payments to S, where (4) is the SECC with some lower bound B.

$$\sum_{i\in S} p_i^{VN} \geq B \tag{4}$$

$$p_i^{VN} \leq b_i \quad \text{for } i \in S \tag{5}$$

The quadratic optimization problem to be solved is minimizing the Euclidean distance between p^{VN} and p^V under the constraints above. For the solution of this optimization problem, the voluntary participation constraint (5) will be active for some i. Let A be the set of indices for which (5) is active, i.e. $p_i^{VN} = b_i$ for $i \in A$.

For the remaining $i \in S \setminus A$, we write $p_i^{VN} = p_i^V + \delta_i$. The single effective core constraint (4) can now be rewritten as

$$\sum_{i\in S\setminus A} \delta_i \geq B - \sum_{i\in S\setminus A} p_i^V - \sum_{i\in A} b_i.$$

Minimizing the Euclidean distance between p^{VN} and p^V is equivalent to minimizing $\sum_{i\in S\setminus A} \delta_i^2$. Since we have a lower bound on the sum of the δ_i, the minimum possible value of $\sum_{i\in S\setminus A} \delta_i^2$ is achieved when all δ_i are equal, i.e.

$$\delta_i = \delta = \frac{1}{|S\setminus A|}\left(B - \sum_{j\in S\setminus A} p_j^V - \sum_{j\in A} b_j\right) \tag{6}$$

for $i \in S \setminus A$. With that we conclude

$$p_i^{VN} = \begin{cases} b_i & \text{for } i \in A \\ p_i^V + \delta & \text{for } i \in S\setminus A. \end{cases} \tag{7}$$

Finally, we verify that VN is non-decreasing. Assume bidder i increases their bid and this does not change the allocation x. If i is a losing bidder in x, their VN payment is 0 and can obviously not decrease. Furthermore, if i is a winning bidder, but i's payment is not part of the SECC, i's VN payment will equal their VCG payment which does not change as it only depends on the other bids. From now on, we assume that bidder i is a winning bidder whose payment is part of the SECC, i.e. $i \in S$.

Consider how the quadratic optimization problem changes when increasing bidder i's bid. One constraint and the point p^V move continuously with this change. So clearly the solution, i.e. p^{VN}, also moves continuously. During this move some of the constraints (5) will become active or inactive. We call the moments when this happens *switches* and examine the *steps* between two consecutive switches.

As p^{VN} changes continuously around switches, equation (7) will yield the same result at the switch, no matter if we consider the switching constraint to be active or not. So for every single step we can assume that the set of active constraints is the same at the beginning and the end of the step. If suffices to show that bidder i's payment does not decrease in every step between two switches. (In short, p^{VN} is a continuous function which is piecewise defined and we verify all of its pieces are non-decreasing.)

Assume bidder i's bid increases from b_i to b'_i in a certain step and let $b = (b_i, b_{-i})$ and $b' = (b'_i, b_{-i})$ denote the corresponding bid profiles. We distinguish two case based on if i is in the set of active constraints in this step or not. If i's constraint is active, i.e. $i \in A$, we have $p_i^{VN}(b, x) = b_i$ and $p_i^{VN}(b', x) = b'_i$ in (7). Then the voluntary participation constraint implies

$$p_i^{VN}(b, x) = b_i \leq b'_i = p_i^{VN}(b', x).$$

Otherwise, for $i \notin A$, we have $p_i^{VN}(b, x) = p_i^V + \delta$ and $p_i^{VN}(b', x) = p_i^V + \delta'$ where δ' is the term in (6) for the bidding profile b' with the increased bid. Then it remains to argue that $\delta \leq \delta'$. This is true since neither B nor $|S \setminus A|$ in (6) change. The sum $\sum_{j \in A} b_j$ also stays the same since $i \notin A$. Furthermore, $\sum_{j \in S \setminus A} p_j^V$ decreases or stays the same because the VCG payments of all other bidders decrease or stay the same when a winning bidder increases their bid. $\square$

The above proves that the existence of an SECC is a sufficient condition for the VN payment to be non-decreasing. It is however, not a necessary condition as the following example in Table 2 shows (see Figure 2 for the example's conflict graph).

Bidder	1	2	3	4	5
Bundle k_i	$\{A\}$	$\{B\}$	$\{C\}$	$\{A, B\}$	$\{A, C\}$

Table 2: An example of an SMCA with more than a single effective core constraint that is non-decreasing.

If the bidders b_1, b_2 and b_3 win the auction, the following two core constraints are effective:

$$p_1 + p_2 \geq b_4$$
$$p_1 + p_3 \geq b_5$$

The remaining core constraints are already covered by these two constraints above. However in general, none of the two fully covers the other, meaning there is more than one effective core constraint. On the other hand, this example is in fact non-decreasing. To prove this, we briefly discuss two necessary conditions for when overbidding can occur. Firstly, increasing bid b_i must decrease the VCG payment p_j^V of another winner. This is clear since increasing b_i will not change p_i^V and cannot increase another winner's VCG payment. Furthermore, if the VCG payment does not change, the VCG nearest clearly will not either.

Secondly, the decrease of p_j^V must move the VCG nearest point, and not only decrease p_j^{VN}, but also p_i^{VN}. The VCG nearest point will move along a number of faces of the core. Each of the faces if defined by a subset of core constraints being tight. During the movement along at least one of these faces, p_i^{VN} must decrease.

In this example, there are only the two core constraints $p_1 + p_2 \geq b_4$ and $p_1 + p_3 \geq b_5$. So the only face to consider is the line defined by both constraints being tight. This can be parametrized as

$$\begin{pmatrix} 0 \\ b_4 \\ b_5 \end{pmatrix} + x \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

with $x \in \mathbb{R}$. Note that only the second and third entry of the directional vector have the same sign. Hence, 2 and 3 are the only potential indices j such that a decrease of

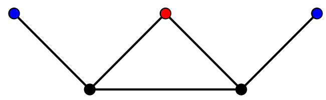

Figure 2: The conflict graph of the auction in Table 2. The bidders $4, 5$ are colored in black. The bidders $2, 3$ are colored in blue. The bidder 1 is colored in red.

p_j^V could cause p_j^{VN} and p_i^{VN} decrease. However, the VCG payments are

$$\begin{aligned} p_1^V &= \max(b_4 - b_2, b_5 - b_3) \\ p_2^V &= \max(b_4 - b_1, 0) \\ p_3^V &= \max(b_5 - b_1, 0). \end{aligned}$$

In particular, increasing b_2 or b_3 will not decrease p_3^V and p_2^V, respectively. So no bidder can decrease their payment by increasing their bid.

5. Graph Representation of Auction Classes

In what follows, we focus on identifying auction categories for which an SECC is assured to exist. For this purpose, we employ graph-based models to depict these auction classes. Specifically, we construct what we term as a *conflict graph*, which captures the overlapping interests in the auction bundles.

Consider an interest profile $k = (k_1, \ldots, k_n)$ for a given SMCA. We define a graph $G = (V, E)$, where the vertex set V consists of the elements of k, that is, $V = \{k_1, \ldots, k_n\}$. Each vertex in this graph symbolizes a particular bidder. An edge between two vertices is formed if only if the corresponding bundles have at least one shared item. Figure 3 showcases two straightforward examples of such conflict graphs.

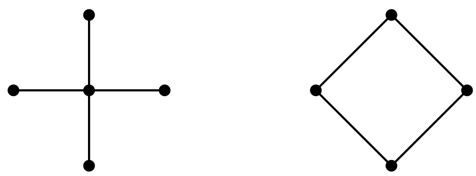

Figure 3: Two examples of conflict graphs. The left one corresponds to the interest profile $(\{A\}, \{B\}, \{C\}, \{D\}, \{A, B, C, D\})$, the right one to $(\{A, B\}, \{B, C\}, \{C, D\}, \{D, A\})$.

In the conflict graph, every set of winners in the auction corresponds to a maximal independent set (MIS).

It is noteworthy that any graph having n vertices can occur as the conflict graph for an SMCA with n participating bidders. In other words, this relationship is surjective. To illustrate, given a specific graph, link each edge to a distinct auction item. Then, for each vertex, choose the associated bundle that comprises the linked items of all adjacent edges.

Even though various interest profiles could yield an identical conflict graph, auctions sharing the same conflict graph inherently produce equivalent core constraints.

Lemma 5.1. *Interest profiles with the same conflict graph have equivalent core constraints (up to renaming the bidders) for all possible bid profiles.*

Proof. Consider two interest profiles that yield conflict graphs which are isomorphic. We can re-label the bidders in one of the profiles so that the isomorphism maps the i-th bidder of one graph to the i-th bidder in the other graph, for all i in $\{1, \ldots, n\}$.

Recall Definition 3.1 of core constraints:

$$\sum_{i \in N \setminus L} p_i(b,x) \geq W(b, X(b_L)) - W(b, x_L)$$

Note that for any set L, both x_L and $X(b_L)$ are solely functions of the conflict graphs and the associated bid profile. Consequently, this extends to the entire right-hand side of the inequality, establishing that the core constraints are indeed equivalent. □

By examining the conflict graph alone, we can determine whether an SECC exists based on the following sufficient conditions.

Lemma 5.2. *An auction with a conflict graph that is a complete multipartite graph has a SECC.*

Proof. Assume the conflict graph is a complete multipartite graph. In this case, bidders can be partitioned into k distinct groups, $B_1, \dots, B_k$, such that no two bidders within the same group are connected by an edge, but all bidders from different groups are connected.

We assert that a winning set must be a single bidder group. Indeed, bidders from multiple groups cannot be in a winning set due to overlapping bundles. Additionally, if the winning set contains only a subset of a bidder group, the allocation is suboptimal: the remaining bidders in the group could be included to enhance social welfare.

Let B_w be the winning bidder group. We now argue that there exists only one effective core constraint. For any subset $L \subseteq N$, the core constraint is given by:

$$\sum_{i \in N \setminus L} p_i(b,x) \geq W(b, X(b_L)) - W(b, x_L). \tag{8}$$

First, observe that if $N \setminus L$ includes a losing bidder, the constraint is ineffective. Including this losing bidder in L does not alter the left-hand side of 8 since the losing bidder's payment is zero. but makes the right-hand side no worse, hence the new constraint subsumes the old one.

Thus, we only need to consider the core constraints with $N \setminus L \subset B_w$. Choose L' such that $B_w \setminus L' = N \setminus L$. Furthermore, let B_o be the winning bidder group in the auction with only the bidders $N \setminus B_w$. The term $W(b, X(b_L))$ on the right-hand side equals either $\sum_{i \in L'} b_i$ or $\sum_{i \in B_o} b_i$.

If it is the former, the right-hand side is zero, rendering the constraint ineffective. In the latter case, we get a constraint of the form:

$$\sum_{i \in B_w \setminus L'} p_i(b,x) \geq \sum_{i \in B_o} b_i - \sum_{i \in L'} b_i.$$

As $p_i(b,x) \leq b_i$, all such constraints are subsumed by

$$\sum_{i \in B_w} p_i(b,x) \geq \sum_{i \in B_o} b_i$$

which proves that this is the sole effective core constraint. □

Note that both graphs in Figure 3 are complete bipartite, signifying that an SECC exists for any auctions represented by such conflict graphs. We present another sufficient condition for the existence of an SECC below.

Lemma 5.3. *If every MIS in the conflict graph contains at most two nodes, then the auction is guaranteed to have an SECC.*

Proof. Recall from the proof of Lemma 5.2 that it is sufficient to focus on core constraints, where $N \setminus L$ contains only winning bidders. In such a case, we arrive at constraints of the form $p_i + p_j \geq B$, along with individual constraints for p_i and p_j. These individual constraints can be either $p_i \geq 0$ or $p_i \geq B - b_j$ (and similarly for p_j).

Given that $p_i \leq b_i$ and $p_j \leq b_j$, it follows that $p_i + p_j \geq B$ is the only effective core constraint, thereby affirming the lemma.

□

The Lemmas 5.2 and 5.3 provide two sufficient conditions under which an SECC is guaranteed to exist. It is crucial to note, however, that these conditions are not necessary for the existence of an SECC. This is demonstrated by the example detailed in Table 3 and its associated conflict graph depicted in Figure 4. Despite the fact that the conflict graph has a MIS larger than two and is not a complete multipartite graph, it still exhibits an SECC.

Lemma 5.4. *There exists an SMCA that is not a complete multipartite graph which has an SECC and has an MIS of size larger than 2.*

Proof. Consider the SMCA with the interest profile shown in Table 3.

Bidder	1	2	3	4	5
Bundle k_i	$\{A, B\}$	$\{B, C\}$	$\{C, D\}$	$\{D, A, E\}$	$\{E\}$

Table 3: An example of an SMCA that has an SECC, where the conflict graph has a MIS of size larger than 2 and is not a complete multipartite graph.

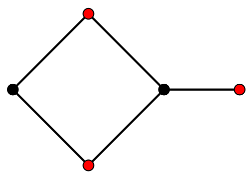

Figure 4: The conflict graph of the auction in Table 3. The MIS with three nodes (bidders $1, 3, 5$) is colored in red.

We start by addressing the case where two or fewer bidders win. Here, Lemma 5.3 already guarantees the existence of an SECC.

The only situation where at least three bidders win is when bidders 1, 3, and 5 are the winners. For this scenario, the core constraints on the payments of the winners are as follows:

$$p_1 + p_3 + p_5 \geq b_2 + b_4 \tag{CC1}$$

$$p_1 + p_5 \geq \max((b_2 + b_4), b_3) - b_3 \tag{CC2}$$

$$p_3 + p_5 \geq \max((b_2 + b_4), b_1) - b_1 \tag{CC3}$$

$$p_1 + p_3 \geq \max((b_2 + b_4), (b_2 + b_5)) - b_5 \tag{CC4}$$

In the following, we argue that (CC1) is the only effective core constraint since it covers all other constraints.

We first demonstrate that (CC2) and (CC3) are never effective core constraints. If (CC2) is non-trivial, then we have $p_1 + p_5 \geq b_2 + b_4 - b_3$. Since $p_3 \leq b_3$, (CC2) is satisfied as soon as the payments of winners satisfies (CC1). Therefore, (CC2) is never an effective core constraint. A similar argument applies to (CC3) due to symmetry.

We now show that the (CC1) is always a tight core constraint of any VCG-nearest payment, i.e., $p_1 + p_3 + p_5 = b_2 + b_4$. We prove this statement by showing that the following three scenarios are impossible.

The first scenario is that neither (CC1) nor (CC4) is tight for the VCG-nearest point. In such a scenario, the payment must be equal to the VCG payment, as those are the remaining constraints on every individual winning bidders payments. Since (CC1) is not tight, we have $p_1^V + p_3^V + p_5^V > b_2 + b_4$ which implies

$$\begin{aligned}
&\max\big((b_2+b_4),(b_1+b_3),(b_2+b_5)\big)\\
&+\max\big((b_1+b_5),(b_2+b_4),(b_2+b_5)\big)\\
&+\max\big((b_2+b_4),(b_1+b_3)\big) > b_2+b_4+2(b_1+b_3+b_5).
\end{aligned}$$

If $(b_2 + b_4)$ is maximal in any of the three items, we see that the above inequality cannot hold, due to the fact that bidders $1, 3, 5$ winning which means $\max(b_2 + b_4, b_2 + b_5) \leq b_1 + b_3 + b_5$. Therefore, we obtain the following expression,

$$\begin{aligned}
\max\big((b_3+b_5),(b_2+b_5)\big)+\max\big((b_1+b_5),(b_2+b_5)\big)+(b_1+b_3)&\\
> b_2+b_4+2(b_1+b_3+b_5)&
\end{aligned}$$

However, all possible outcomes of the left side are strictly inferior to the right side. Therefore, $p_1^V + p_3^V + p_5^V \leq b_2 + b_4$, which contradicts to our assumption for the first scenario that (CC1) is not tight.

The second scenario is that (CC4) is tight and (CC1) is not tight. Then we have either $p_1 + p_3 = b_2 + b_4 - b_5$, or $p_1 + p_3 = b_2$.

If $p_1 + p_3 = b_2 + b_4 - b_5$, then $p_1 + p_3 + p_5 \leq b_2 + b_4$ because $p_5 \leq b_5$. This implies that (CC1) is tight, which contradicts to our assumption of the second scenario.

If $p_1 + p_3 = b_2$, we have $p_5 > b_4$ because (CC1) is not tight. Then we know that p_5 equals the VCG payment p_5^V, as it is the remaining constraint on bidder 5's winning payment. However, $p_5^V > b_4$ implies that,

$$\begin{aligned}
\max((b_2+b_4),(b_1+b_3))-(b_1+b_3) &> b_4\\
b_2+b_5 &> b_1+b_3+b_5
\end{aligned}$$

This contradicts the assumption that bidders $1, 3, 5$ are winning. Therefore, (CC1) is a tight constraint for all VCG-nearest payment in the SMCA.

The next step of the proof is to show that (CC4) is always covered by (CC1). If $b_2 + b_4 > b_2 + b_5$, because $b_5 \geq p_5$, it is trivial that (CC4) always satisfies when (CC1) satisfies. Otherwise, if $b_2 + b_5 > b_2 + b_4$, we subtract (CC4) $p_1 + p_3 \geq b_2$ from (CC1) $p_1 + p_3 + p_5 = b_2 + b_4$ (as (CC1) is always tight). Then we have $p_5 \leq b_4$, which indicates that if (CC1) holds, then (CC4) also holds.

Hence, (CC1) is the only effective core constraint in the SMCA. □

The results so far show that examining the conflict graph provides insights into the auction's structure, particularly into whether an SECC exists. Consequently, we are able to tell that certain payment rules are non-decreasing for a particular auction. On the other hand, we are also able to identify scenarios in which the non-decreasing property is violated: By examining induced subgraphs of the conflict graph, we are able to determine that the non-decreasing property of a payments rule is violated for a particular auction. This is formalized in the following lemma.

Lemma 5.5. *Consider two interest profiles k and k' with corresponding conflict graphs G and G'. If G' is an induced subgraph of G and a payment rule is not non-decreasing for k', then the payment rule is also not non-decreasing for k.*

Proof. According to Definition 3.5, a payment rule not being non-decreasing for k' means there exists an allocation x and bid profiles b and b' with $b'_i \geq b_i$ and $b_{-i} = b'_{-i}$ such that $p_i(b', x) < p_i(b, x)$. By simply choosing zero (or arbitrarily small) bids for all bidders in $G \setminus G'$, we also find two bid profiles with the same property for k. □

Bosshard et al. demonstrated that VN payments can violate the non-decreasing property by presenting a specific interest profile along with corresponding bids [11]. Consequently, VN payments can also be expected to fail the non-decreasing test in any auction whose conflict graph contains the example graph from Bosshard et al. as an induced subgraph. This realization underscores the importance of identifying minimal examples where overbidding occurs, and it provides a strong impetus for the discovery of additional sufficient conditions under which overbidding is precluded. In what follows, we outline a novel sufficient condition that ensures the non-decreasing property of VN payments, independent of the existence of an SECC.

Theorem 5.6. *For SMCAs characterized by an interest profile in which each winning allocation includes no more than three winning bidders, the VN payment mechanism exhibits non-decreasing behavior.*

Proof. The case that the auction is won by two bidders is already treated in Lemma 5.3. Assume three bidders win the auction, and without loss of generality, let the winners be bidders 1, 2 and 3. Then the core constraints are

$$p_1^{VN} + p_2^{VN} + p_3^{VN} \geq W(b, X(b_{N\setminus\{1,2,3\}})) \tag{9}$$

$$p_1^{VN} + p_2^{VN} \geq W(b, X(b_{N\setminus\{1,2\}})) - b_3 \tag{10}$$

$$p_2^{VN} + p_3^{VN} \geq W(b, X(b_{N\setminus\{2,3\}})) - b_1 \tag{11}$$

$$p_1^{VN} + p_3^{VN} \geq W(b, X(b_{N\setminus\{1,3\}})) - b_2. \tag{12}$$

Remember, that we can ignore core constraints of the form $p_i^{VN} \geq p_i^V$ (VCG-constraints). Furthermore, assume without loss of generality that bidder 3 increases their bid.

Let M be the minimum revenue determined by the core constraints. There are two possibilities for the minimum revenue core: First, if the plane described by (9) is not fully covered by the constraints (10), (11) and (12), the minimum revenue is $M = W(b, X(b_{N\setminus\{1,2,3\}}))$. We further discuss this case in the next paragraph. The second possibility is that the plane described by (9) is fully covered by the other constraints, and $M > (b, X(b_{N\setminus\{1,2,3\}}))$. Then the minimum revenue core is a single point determined by equality holding in (10), (11) and (12). Since the right sides of (10), (11) and (12) are not larger than the right side of (9), all three constraints are needed to fully cover the plain. In particular, bidder 3 must be part of $X(b_{N\setminus\{1,2\}})$, otherwise (9) implies (10), and the plane is not fully covered. But this means, that increasing b_3 does not change the right sides of (10). As the same is true for (11) and (12), increasing b_3 does not move the minimum revenue core and thereby the VN payment point.

In the following, we assume that constraint (9) is active, and $M = W(b, X(b_{N\setminus\{1,2,3\}}))$. We argue similarly to the proof of Theorem 4.2: All changes in the VN payments are continuous in the change of the bid b_3. At any time, a number of constraints are active, and this set of active constraints changes at certain switches. To prove, the payment does not decrease overall, it suffices to prove it does not decrease between two switches, when the set of active constraints does not change. In the following, we distinguish three possible cases.

1st case. Only constraint (9) is active. Hence, the VN payments are

$$\begin{pmatrix} p_1^V + \frac{M-(p_1^V+p_2^V+p_3^V)}{3} \\ p_2^V + \frac{M-(p_1^V+p_2^V+p_3^V)}{3} \\ p_3^V + \frac{M-(p_1^V+p_2^V+p_3^V)}{3} \end{pmatrix}.$$

When b_3 is increased, the minimum revenue $M = W(b, X(b_{N\setminus\{1,2,3\}}))$ does not change. Furthermore, p_1^V and p_2^V stay the same or decrease. So bidder 3's payment does not decrease according to the formula above.

2nd case. The constraints (9) and (10) are active. This implies

$$\begin{aligned} p_3^{VN} &= b_3 + M - W(b, X(b_{N\setminus\{1,2\}})) \\ p_1^{VN} + p_2^{VN} &= W(b, X(b_{N\setminus\{1,2\}})) - b_3. \end{aligned}$$

As b_3 increases, $W(b, X(b_{N\setminus\{1,2\}}))$ can increase by at most as much as b_3. Hence, the payment p_3^{VN} will not decrease.

3rd case. Constraints (9) and (11) are active. (Note that the case when constraints (9) and (12) are active is equivalent due to symmetry.) This implies

$$\begin{aligned} p_1^{VN} &= b_1 + M - W(b, X(b_{N\setminus\{2,3\}})) \\ p_2^{VN} + p_3^{VN} &= W(b, X(b_{N\setminus\{2,3\}})) - b_1. \end{aligned}$$

These equations describe a line on which the VN payment points lies. Increasing b_3 does not change the right side of the equation. Furthermore, it may decrease p_2^V, but does not change p_3^V. Since p^{VN} is the closest point to p^V on the line, this can, if it causes a change, only lead to a decrease of p_2^{VN} and an increase of p_3^{VN}. □

Corollary 5.7. *For any SMCA involving no more than five participants, the VN payment mechanism maintains a non-decreasing property.*

Proof. When the auction features up to three winners, we can directly apply Theorem 5.6 to demonstrate the non-decreasing nature of the VN payment scheme.

In scenarios with either four participants and four winners or five participants and five winners, the corresponding conflict graphs stand disconnected. In such instances, these auctions effectively break down into smaller, self-contained sub-auctions. Consequently, the VN payment rule remains non-decreasing by virtue of the partitioned conflict graphs.

Lastly, we consider the case where an auction includes five bidders but only four emerge as winners. In this situation, the only feasible connected conflict graph is $K_{1,4}$, a specific type of complete bipartite graph. This case has been adequately addressed by Lemma 5.2, thereby confirming the non-decreasing character of the VN payment rule here as well. □

Concluding the section, we conjecture that the over bidding example with six bidders given in [11] is actually minimal with respect to the number of winners.

Conjecture 5.8. *The VN-payment rule is non-decreasing for all auctions that have an interest profile for which every winner allocation contains five or fewer winners.*

The remaining gap to Theorem 5.6 is ruling out cases of four and five winners.

In the following, we provide a starting point to prove the conjecture. For this purpose, let us take a closer look at what leads to the overbidding in the example in [11] (see Table 4). In the example, bidder 3 increasing their bid leads to a decreased payment for bidder 3. Before and after this increase, the following core constraints are active (i.e. equality holds):

$$\begin{aligned}
p_1 + p_2 \quad\quad + p_4 \quad\quad\quad\quad &= 5 \\
p_1 \quad\quad + p_3 \quad\quad\quad\quad + p_6 &= 7 \\
p_2 + p_3 + p_4 \quad\quad\quad\quad &= 5 \\
p_2 + p_3 \quad\quad + p_5 \quad\quad &= 5 \\
p_4 + p_5 + p_6 &= 2
\end{aligned}$$

The solutions of this system of equations can be parametrized in p_1 as

$$p = \begin{pmatrix} 0 \\ 5/2 \\ 0 \\ -5/2 \\ -5/2 \\ 7 \end{pmatrix} + p_1 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \\ 1 \\ -2 \end{pmatrix} \tag{13}$$

When bidder 3 increases their bid from 4 to 5, bidder 1's VCG payment decreases from 2 to 1. Since the VN payment minimizes the distance to the VCG payment point, this also leads to bidder 1's VN payment decreasing. And since bidder 3's payment has a positive entry in the directional vector in (13), it decreases with bidder 1's payment.

Hence, we can formulate the following necessary condition for the VN payment to not be non-decreasing. There exist two bidders i and j, such that (1) increasing i's bid decreases j's VCG payment, and (2) i's and j's payments have the same sign as in the parametrized form of the VN payments.

Bidder	Bundle k_i	Bid b_i	VCG	VN	Bid b'_i	VCG'	VN'
1	$\{A\}$	5	2	37/12	5	1	36/12
2	$\{B\}$	5	0	16/12	5	0	18/12
3	$\{C\}$	**4**	1	**37/12**	**5**	1	**36/12**
4	$\{D\}$	1	0	7/12	1	0	6/12
5	$\{E\}$	1	0	7/12	1	0	6/12
6	$\{F\}$	1	0	10/12	1	0	12/12
7	$\{A, B, D\}$	5			5		
8	$\{B, C, E\}$	5			5		
9	$\{A, C, F\}$	7			7		
10	$\{D, E, F\}$	2			2		
11	$\{B, C, D\}$	3			5		

Table 4: Overbidding for 11 bidders and 6 winners, found by Bosshard et al. [11]. By increasing their bid, bidder 3 decreases their payment.

6. Over-bidding for Non-decreasing Payment Rules

In this section, we show that no over-bidding is profitable for non-decreasing payment rules, which proves a conjecture by Bosshard et al. [11].

6.1. Over-bidding on Winning Bids

As long as an over-bid does not change the winner allocation compared to the truthful bid, it will not increase the bidders utility. This follows directly from the definition of the non-decreasing payment rules: Increasing a bid will not decrease the payment. However, the allocated value stays the same since the allocation does not change.

Since any efficient allocation remains efficient when increasing a winning bid, over-bidding when the truthful bid is already a winning bid is not profitable. This implies the following lemma.

Lemma 6.1. *Consider an SMCA with a core-selecting, non-decreasing payment rule. If for a bidder i and fixed bids of the other bidders b_{-i}, the truthful bid v_i is a winning bid, then overbidding does not increase bidder i's utility.*

6.2. Over-bidding on Losing Bids

A losing bid has zero utility due to voluntary participation, i.e., no bundle is acquired, no value is gained, and the payment is zero. A losing over-bid equally results in zero utility. Thus, a profitable over-bidding strategy that increases the utility must result in winning the auction.

The following lemma shows that it is not possible to gain a positive utility by over-bidding, where the truthful bid is losing.

Lemma 6.2. *Consider an SMCA with a core-selecting, non-decreasing payment rule. If for a bidder i and fixed bids of the other bidders b_{-i}, the truthful bid v_i is a losing bid, then over-bidding is not profitable for bidder i.*

Proof. Consider a bidder i who loses when bidding their truthful private value v_i, but wins with an overbid $b_i^o > v_i$. We write the truthful and the over-bidding bid profile as $b^v = (v_i, b_{-i})$ and $b^o = (b_i^o, b_{-i})$, respectively. Furthermore, let x^v and x^o denote the efficient allocations for the bidding profiles b^v and b^o, respectively. Note that $i \notin x^v$, but $i \in x^o$.

The fact that bidder i loses with bid v_i implies that $W(b^v, x^o) < W(b^v, x^v)$. Let $\epsilon = W(b^v, x^v) - W(b^v, x^o)$. We choose $b_i^o = v_i + \epsilon$ as the smallest overbid, such that x^o is an efficient allocation. Then $W(b^o, x^o) = W(b^v, x^v)$. Note that it suffices to consider this overbid since any further increase of the bid beyond this value decreases bidder i's utility according to Lemma 6.1.

We calculate the VCG payment of bidder i for the bidding profile b^o. The maximum reported social welfare without bidder i equals $W(b^o, x^v) = W(b^v, x^v)$ since bidder i loses in x^v. Furthermore, the total reported social welfare of x^o excluding i equals $W(b^o, x^o) - b_i^o$. Therefore,

$$p_i^V(b^o, x^o) = W(b^v, x^v) - (W(b^o, x^o) - b_i^o) = b_i^o.$$

As mentioned in Section 4, a core constraint of the form $p_i(b^o, x^o) \geq p_i^V(b^o, x^o)$ exists for every bidder. But this means that bidder i's payment is at least $b_i^o > v_i$ resulting in a negative utility for bidder i. □

Together Lemmas 6.1 and 6.2 imply the desired result.

Theorem 6.3. *In an SMCA with a core-selecting, non-decreasing payment rule, over-bidding strategy is always weakly dominated by truthful bidding in any Nash equilibrium.*

7. Non-decreasing Property of Other Core-Selecting Payment Rules

We showed in the previous section, that the non-decreasing property ensures that bidders cannot profit from overbidding. However, until now, we have a limited understanding of which payment rules are non-decreasing. In this section, we investigate two common core-selecting payment rules, namely the proxy and the proportional payments, and examine their non-decreasing properties.

Bosshard et al. argued that the proxy and the proportional payments are non-decreasing for multi-minded CAs [11]. However, a counterexample with two bidders and two items A and B exists: Assume $b_1(A) = 12$, $b_1(AB) = 18$ and $b_2(B) = 9$. Under the proportional rule we have that $p_1 = 8$ and $p_2 = 6$. However, if we increase $b_1(A)$ to 15, we get $p_1 = 5$ and $p_2 = 3$. This counterexample occurs because the core constraint for bidder 2 has been lowered by the increase of $b_1(A)$. In this section, we correct the proof of Proposition 2 in [11] for single-minded CAs.

Lemma 7.1. *The proxy payment function and the proportional payment function are non-decreasing for SMCA.*

Proof. Consider a winning bidder i increasing their bid from b_i to b_i'. Because the payment point and the all core constraints move continuously in $\mathbb{R}^n$ with an increasing bid, we can always find a sequence of bids $b_i^0, b_i^1, \ldots, b_i^l$, such that $b_i = b_i^0 < b_i^1 < \ldots < b_i^l = b_i'$ and for the corresponding payment points p^k and p^{k+1} of any two consecutive bids b_i^k and b_i^{k+1} the same core constraints are active.

Now consider two payment points p^k and p^{k+1}. Assume there exists an active core constraint (CC) generated by the set of bidders L, such that $i \notin L$. Then increasing i's bid from b_i^k to b_i^{k+1} does not change the right side of (CC) which is sufficient for the non-decreasing proof in [11]:

Assume that $p_i^{k+1} < p_i^k$. Because $b_i^k < b_i^{k+1}$ this means that the α in the proportional payment's definition must be smaller after the bid increase. In particular, since all other bids are unchanged, this implies $p_j^{k+1} < p_j^k$ for all $j \neq i$. But this contradicts the fact that the sum of payments in the active core constraint does not change. Hence, we must have $p_i^{k+1} \geq p_i^k$.

The argument above also holds for the proxy payment rule, since here too $p_i^{k+1} < p_i^k$ with $b_i^k < b_i^{k+1}$ implies a strictly smaller α after the bid increase. For the proxy payment, this implies $p_j^{k+1} \leq p_j^k$ for all $j \neq i$ which again yields a contradiction, since p_i is part of the left side of (CC).

It now remains to consider the case where $i \in L$ for all core constraints being active at p^k and p^{k+1}. For each such constraint (CC), consider the efficient allocation $X(b_L)$ in the auction with only the bids of bidders in L, and i bidding b_i^k. Similarly, let $X(b_L')$ be the efficient allocation when i bids b_i^{k+1}. Since $b_i^k < b_i^{k+1}$, three scenarios can occur:

1st case. $X(b_L)_i = 0$ and $X(b_L')_i = 0$. Then the right side of (CC) decreases by $b_i^{k+1} - b_i^k$. However, there exists another core constraint (CC_{-i}) which is generated by the set $L \setminus \{i\}$, and is given by

$$\sum_{j \in N \setminus L} p_j(b, x) + p_i^k \geq W(b, X(b_L)) - (W(b, x_L) - b_i^k).$$

The fact that $p_i^k \leq b_i^k$, together with (CC) being active, i.e.

$$\sum_{j \in N \setminus L} p_j(b, x) = W(b, X(b_L)) - W(b, x_L),$$

imply that equality must hold in (CC_{-i}). The same is true for p_i^{k+1}. Therefore, (CC_{-i}) is active for both p^k and p^{k+1}. Furthermore, since $i \notin L \setminus \{i\}$ and (CC_{-i}) includes p_i in its left side, the non-decreasing property follows as argued previously.

2nd case. $X(b_L)_i = 0$ and $X(b'_L)_i = 1$. Then a switching point b_i^* with $b_i^k \le b_i^* \le b_i^{k+1}$ exists, such that i is part of one (but not all) efficient allocations of L with bid b_i^*, but i is not in any efficient allocation with bids smaller than b_i^*. Then we can consider the increases from b_i^k to b_i^* and b_i^* to b_i^{k+1} separately. For the b_i^k to b_i^*, the arguments from the first case apply, while the third case (see below) covers the increase b_i^* to b_i^{k+1}.

3rd case. $X(b_L)_i = 1$ and $X(b'_L)_i = 1$. Then (CC) is not affected by increasing i's bid from b_i^k to b_i^{k+1}. (Note that this argument does not hold in the general multi-minded setting, where the proportional payment has shown to not be non-decreasing.)

We only arrive at this point of the proof, if for all active core constraints, case three applies, which means no active core constraints is changed by the bid increase. In particular, this means that, for both the proportional and proxy payment, α does not decrease with the bid increase. Together with $b_i^k < b_i^{k+1}$ this implies that i's payment does not decrease.

Finally, since i's payment does not decrease between any b_i^k to b_i^{k+1}, it also does not decrease between b_i and b'_i.

□

8. Conclusion

Our research has shed light on the interconnections between core constraints and payment mechanisms for CAs. In specific regard to single-minded CAs, we demonstrate how the presence of a single effective core constraint fortifies the VN payment rule to sustain its non-decreasing property. This, in turn, promotes fair competitive practices among bidders.

In an effort to make these complex relationships more accessible, we have introduced the conflict graph representation for SMCAs. This model serves as an analytical tool from which we have been able to identify sufficient conditions for the existence of a single effective core constraint in the auction. This has broader implications, especially in auction designs that aim to balance efficiency and fairness. As future work in this direction, proving or disproving the conjecture on the minimal overbidding examples remains.

As a final point, we have established that over-bidding will not produce favorable outcomes under any Nash equilibrium settings when the auction is governed by non-decreasing payment rules.

Acknowledgement

This work was supported by grants from the Science and Technology Development Fund (FDCT) Macau SAR (File no. 0129/2022/A, and no. 0078/2023/AMJ) and the University of Macau (File no. MYRG-CRG2022-00013-IOTSC-ICI, and no. SRG2022-00032-FST).

References

[1] S. J. Rassenti, V. L. Smith, R. L. Bulfin, A combinatorial auction mechanism for airport time slot allocation, The Bell Journal of Economics (1982) 402–417.

[2] L. M. Ausubel, O. Baranov, A practical guide to the combinatorial clock auction, The Economic Journal 127 (605) (2017) F334–F350. doi:https://doi.org/10.1111/ecoj.12404.

[3] I. Palacios-Huerta, D. C. Parkes, R. Steinberg, Combinatorial auctions in practice, Available at SSRN 3844338 (2022).

[4] W. Vickrey, Counterspeculation, auctions, and competitive sealed tenders, The Journal of finance 16 (1) (1961) 8–37.

[5] E. H. Clarke, Multipart pricing of public goods, Public choice 11 (1) (1971) 17–33.

[6] T. Groves, Incentives in teams, Econometrica: Journal of the Econometric Society (1973) 617–631.

[7] L. M. Ausubel, P. Milgrom, et al., The lovely but lonely vickrey auction, Combinatorial auctions 17 (2006) 22–26.

[8] R. W. Day, P. Cramton, Quadratic core-selecting payment rules for combinatorial auctions, Operations Research 60 (3) (2012) 588–603.

[9] R. W. Day, S. Raghavan, Fair payments for efficient allocations in public sector combinatorial auctions, Management science 53 (9) (2007) 1389–1406.
URL https://pubsonline.informs.org/doi/pdf/10.1287/mnsc.1060.0662

[10] R. Day, P. Milgrom, Core-selecting package auctions, international Journal of game Theory 36 (3-4) (2008) 393–407.
URL https://link.springer.com/article/10.1007/s00182-007-0100-7

[11] V. Bosshard, Y. Wang, S. Seuken, Non-decreasing payment rules for combinatorial auctions., in: IJCAI, 2018, pp. 105–113.

[12] J. K. Goeree, Y. Lien, On the impossibility of core-selecting auctions, Theoretical economics 11 (1) (2016) 41–52.

[13] L. M. Ausubel, O. Baranov, Core-selecting auctions with incomplete information, International Journal of Game Theory 49 (1) (2020) 251–273.

[14] E. Markakis, A. Tsikiridis, On core-selecting and core-competitive mechanisms for binary single-parameter auctions, in: International Conference on Web and Internet Economics, Springer, 2019, pp. 271–285.

[15] B. Bünz, S. Seuken, B. Lubin, A faster core constraint generation algorithm for combinatorial auctions, Proceedings of the AAAI Conference on Artificial Intelligence 29 (1) (Feb. 2015). doi:10.1609/aaai.v29i1.9289.
URL https://ojs.aaai.org/index.php/AAAI/article/view/9289

[16] R. Sano, Incentives in core-selecting auctions with single-minded bidders, Games and Economic Behavior 72 (2) (2011) 602–606.

[17] M. Ott, M. Beck, Incentives for overbidding in minimum-revenue core-selecting auctions, no. F16-V3 in Beiträge zur Jahrestagung des Vereins für Socialpolitik 2013: Wettbewerbspolitik und Regulierung in einer globalen Wirtschaftsordnung - Session: Auctions and Licensing, ZBW - Deutsche Zentralbibliothek für Wirtschaftswissenschaften, Leibniz-Informationszentrum Wirtschaft, Kiel und Hamburg, 2013.
URL http://hdl.handle.net/10419/79946