SINR Diagrams:
Convexity and its Applications in Wireless Networks

Chen Avin* Yuval Emek† Erez Kantor‡ Zvi Lotker* David Peleg§
Liam Roditty¶

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Abstract

The rules governing the availability and quality of connections in a wireless network are described by *physical* models such as the *signal-to-interference & noise ratio (SINR)* model. For a collection of simultaneously transmitting stations in the plane, it is possible to identify a *reception zone* for each station, consisting of the points where its transmission is received correctly. The resulting *SINR diagram* partitions the plane into a reception zone per station and the remaining plane where no station can be heard.

SINR diagrams appear to be fundamental to understanding the behavior of wireless networks, and may play a key role in the development of suitable algorithms for such networks, analogous perhaps to the role played by Voronoi diagrams in the study of proximity queries and related issues in computational geometry. So far, however, the properties of SINR diagrams have not been studied systematically, and most algorithmic studies in wireless networking rely on simplified *graph-based* models such as the *unit disk graph (UDG)* model, which conveniently abstract away interference-related complications, and make it easier to handle algorithmic issues, but consequently fail to capture accurately some important aspects of wireless networks.

The current paper focuses on obtaining some basic understanding of SINR diagrams, their properties and their usability in algorithmic applications. Specifically, we have shown that assuming uniform power transmissions, the reception zones are convex and relatively well-rounded. These results are then used to develop an efficient approximation algorithm for a fundamental point location problem in wireless networks.

*Department of Communication Systems Engineering, Ben Gurion University, Beer-Sheva, Israel. E-mail:{avin,zvilo}@cse.bgu.ac.il. Zvi Lotker was partially supported by a gift from Cisco research center and the Israel Science Foundation (grant no. 894/09).

†Computer Engineering and Networks Laboratory (TIK), ETH Zurich, Zurich, Switzerland. E-mail: yuval.emek@tik.ee.ethz.ch.

‡Department of Electrical Engineering, Technion, Haifa, Israel. E-mail: erez.kantor@ee.technion.ac.il.

§Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, Rehovot, Israel. E-mail: david.peleg@weizmann.ac.il. Supported in part by grants from the Minerva Foundation, the Israel Ministry of Science and the Israel Science Foundation (grant no. 894/09).

¶Department of Computer Science, Bar Ilan University, Ramat-Gan, Israel. E-mail: liam.roditty@gmail.com.

1 Introduction

1.1 Background

It is commonly accepted that traditional (wired, point-to-point) communication networks are satisfactorily represented using a graph based model. The question of whether a station s is able to transmit a message to another station s' depends on a single (necessary and sufficient) condition, namely, that there be a wire connecting the two stations. This condition is thus independent of the locations of the two stations, of their other connections and activities, and of the locations, connections or activities of other nearby stations[1].

In contrast, wireless networks are considerably harder to represent faithfully, due to the fact that deciding whether a transmission by a station s is successfully received by another station s' is nontrivial, and depends on the positioning and activities of s and s', as well as on the positioning and activities of other nearby stations, which might interfere with the transmission and prevent its successful reception. Thus such a transmission from s may reach s' under certain circumstances but fail to reach it under other circumstances. Moreover, the question is not entirely "binary", in the sense that connections can be of varying quality and capacity.

The rules governing the availability and quality of wireless connections can be described by *physical* or *fading channel* models (cf. [21, 6, 22]). Among those, the most commonly studied is the *signal-to-interference & noise ratio (SINR)* model. In the SINR model, the energy of a signal fades with the distance to the power of the *path-loss parameter* α. If the signal strength received by a device divided by the interfering strength of other simultaneous transmissions (plus the fixed *background noise* N) is above some *reception threshold* β, then the receiver successfully receives the message, otherwise it does not. Formally, denote by $\text{dist}(p, q)$ the Euclidean distance between p and q, and assume that each station s_i transmits with power ψ_i. (A *uniform power network* is one in which all stations transmit with the same power.) At an arbitrary point p, the transmission of station s_i is correctly received if

$$\frac{\psi_i \cdot \text{dist}(p, s_i)^{-\alpha}}{N + \sum_{j \neq i} \psi_j \cdot \text{dist}(p, s_j)^{-\alpha}} \;\geq\; \beta \;.$$

Hence for a collection $S = \{s_0, \ldots, s_{n-1}\}$ of simultaneously transmitting stations in the d-dimensional space, it is possible to identify with each station s_i a *reception zone* $\mathcal{H}_i$ consisting of the points where the transmission of s_i is received correctly. The common belief is that the path-loss parameter falls in the range $2 \leq \alpha \leq 4$, while the reception threshold is $\beta \approx 6$ (β is always assumed to be greater than 1).

To illustrate how reception depends on the locations and activities of other stations, consider (the numerically generated) Figure 1. Figure 1(A) depicts uniform stations s_1, s_2, s_3 and their

[1] Broadcast domain wired networks such as LANs are an exception, but even most LANs are collections of point-to-point connections.

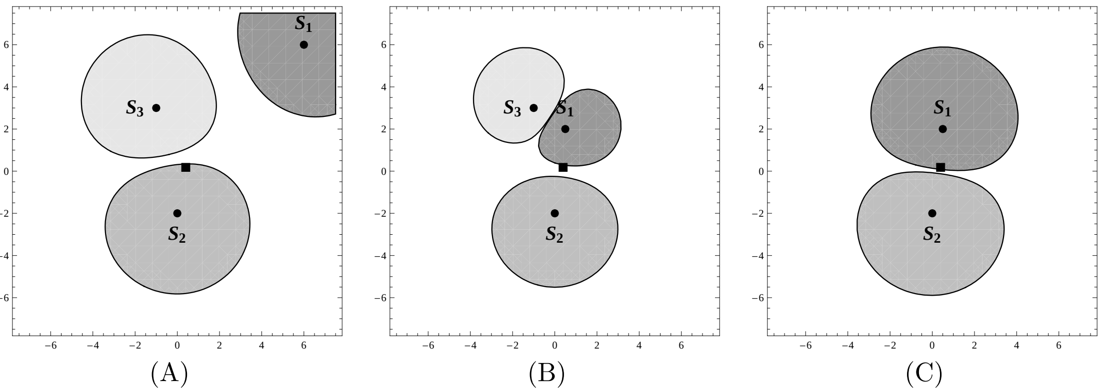


Figure 1: An example of SINR diagram with three transmitters s_1, s_2, s_3 and one receiver denoted by the solid black square. (A) The receiver can hear s_2. (B) Station s_1 moves and now the receiver cannot hear any transmission. (C) If, at the same locations as in (B), s_3 is silent, then the receiver can hear s_1.

reception zones. Point p (represented as a solid black square) falls inside $\mathcal{H}_2$. Figure 1(B) depicts the same stations except station s_1 has moved closer, so that now p does not receive any transmission. Figure 1(C) depicts the stations in the same positions as Figure 1(B), but now s_3 is silent, and as a result, the other two stations have larger reception zones, and p receives the message of s_1.

Figure 1 illustrates a concept central to this paper, namely, the *SINR diagram*. An SINR diagram is a "reception map" characterizing the reception zones of the stations, namely, partitioning the plane into n reception zones $\mathcal{H}_i$, $0 \leq i \leq n-1$, and a zone $\mathcal{H}_\emptyset$ where no station can be heard. In many scenarios the diagram changes dynamically with time, as the stations may choose to transmit or keep silent, adjust their transmission power level, or even change their location from time to time.

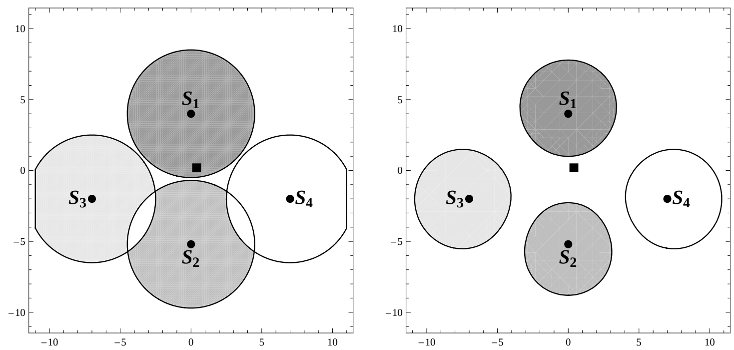
Notice that the SINR diagram adds no new information concerning the locations in which the stations themselves are positioned, since it follows from the definition (refer to Section 2.2 for a formal definition) that station s_i cannot receive the transmission of any station s_j, $j \neq i$, unless the two stations coincide. However, SINR diagrams can be extremely useful for a listening device that does not belong to S and is located at an arbitrary point p in the plane. Using the SINR diagram, it is possible to decide which of the stations of S (if any) can be correctly received at the location p of the listening device.

It is our belief that SINR diagrams are fundamental to understanding the dynamics of wireless networks, and will play a key role in the development of suitable (sequential or distributed) algorithms for such networks, analogous perhaps to the role played by Voronoi diagrams in the study of proximity queries and related issues in computational geometry. Yet, to the best of our knowledge, SINR diagrams have not been studied systematically so far, from either geometric, combinatorial, or algorithmic standpoints. In particular, in the SINR model it is not clear what shapes the reception zones may take, and it is not easy to construct an SINR diagram even in a static setting.

Taking a broader perspective, a closely related concern motivating this paper is that while a fair amount of research exists on the SINR model and other variants of the physical model, little has been done in such models in the *algorithmic* arena. (Some recent exceptions are [10, 11, 12, 13, 14, 15, 17, 19, 20, 23].) The main reason for this is that SINR models are complex and hard to work with. In these models it is even hard to decide some of the most elementary questions on a given setting, and it is definitely more difficult to develop communication or design protocols, prove their correctness and analyze their efficiency.

Subsequently, most studies of higher-layer concepts in wireless multi-hop networking, including issues such as transmission scheduling, frequency allocation, topology control, connectivity maintenance, routing, and related design and communication tasks, rely on simplified *graph-based* models rather than on the SINR model. Graph-based models represent the network by a graph $G = (S, E)$ such that a station s will successfully receive a message transmitted by a station s' if and only if s and s' are neighbors in G and s does not have a concurrently transmitting neighbor in G. In particular, when the stations are deployed in the Euclidean plane, the model of choice for many protocol designers is the *unit disk graph (UDG)* model [8]. In this model, also known as the *protocol* model [12], the transmission of a station can be received by every other station within a unit ball around it. The *UDG graph* is thus a graph whose vertices correspond to the stations, with an edge connecting any two vertices whose corresponding stations are at distance at most one from each other.

Graph-based models are attractive for higher-layer protocol design, as they conveniently abstract away interference-related complications. Issues of topology control, scheduling and allocation are also handled more directly, since notions such as adjacency and overlap are easier to define and test, in turn making it simpler to employ also some useful derived concepts such as domination, independence, clusters, and so on. (Note also that the SINR model in itself is rather simplistic, as it assumes perfectly isotropic antennas and ignores environmental obstructions. These issues can be integrated into the basic SINR model, at the cost of yielding relatively complicated "SINR+" models, even harder to use by protocol designers. In contrast, graph-based models naturally incorporate both directional antennas and terrain obstructions.) On the down side, it should be realized that graph-based models, and in particular the UDG model, ignore or do not accurately capture a number of important physical aspects of real wireless networks. In particular, such models oversimplify the physical laws of interference; in reality, several nodes slightly outside the reception range of a receiver station v (which consequently are not adjacent to v in the UDG graph) might still generate enough cumulative interference to prevent v from successfully receiving a message from a sender station adjacent to it in the UDG graph; see Figure 2 for an example. Hence the UDG model might yield a “false positive” indication of reception. Conversely, a simultaneous transmission by two or more neighbors should not always end in collision and loss of the message; in reality this depends



(A) UDG Diagram (B) SINR Diagram

Figure 2: Interference in the UDG and SINR models. (A) UDG diagram: p can hear s_1. (B) SINR diagram: the cumulative interference of stations s_2, s_3, s_4 prevents p from hearing s_1.

on other factors, such as the relative distances and the relative strength of the transmissions. We illustrate some of these scenarios in Figures 3 and 4, which compare the reception zones of the UDG and SINR models with four transmitting stations s_1, s_2, s_3, s_4 and one receiver p (represented as a solid black square). In the initial setting depicted in Figure 3, only station s_1 transmits, and all others remain silent, so the UDG and SINR diagrams are the same and p can hear s_1 in both models. Figure 4 illustrates three steps of gradually adding s_2, s_3 and s_4 to the transmitting set. When both s_1, s_2 transmit simultaneously, p cannot hear any station in the UDG model, but it does hear s_1 in the SINR model (cases (A) and (B) respectively). Hence in this case the UDG model yields a "false negative" indication. When s_3 joins the transmitting stations, p still cannot hear any station in the UDG model, but now it can hear station s_3 in the SINR model (cases (C) and (D)). When s_4 transmits as well, the effect varies again across the two models (cases (E) and (F)).

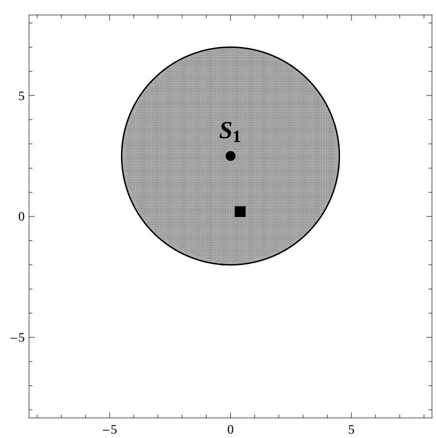
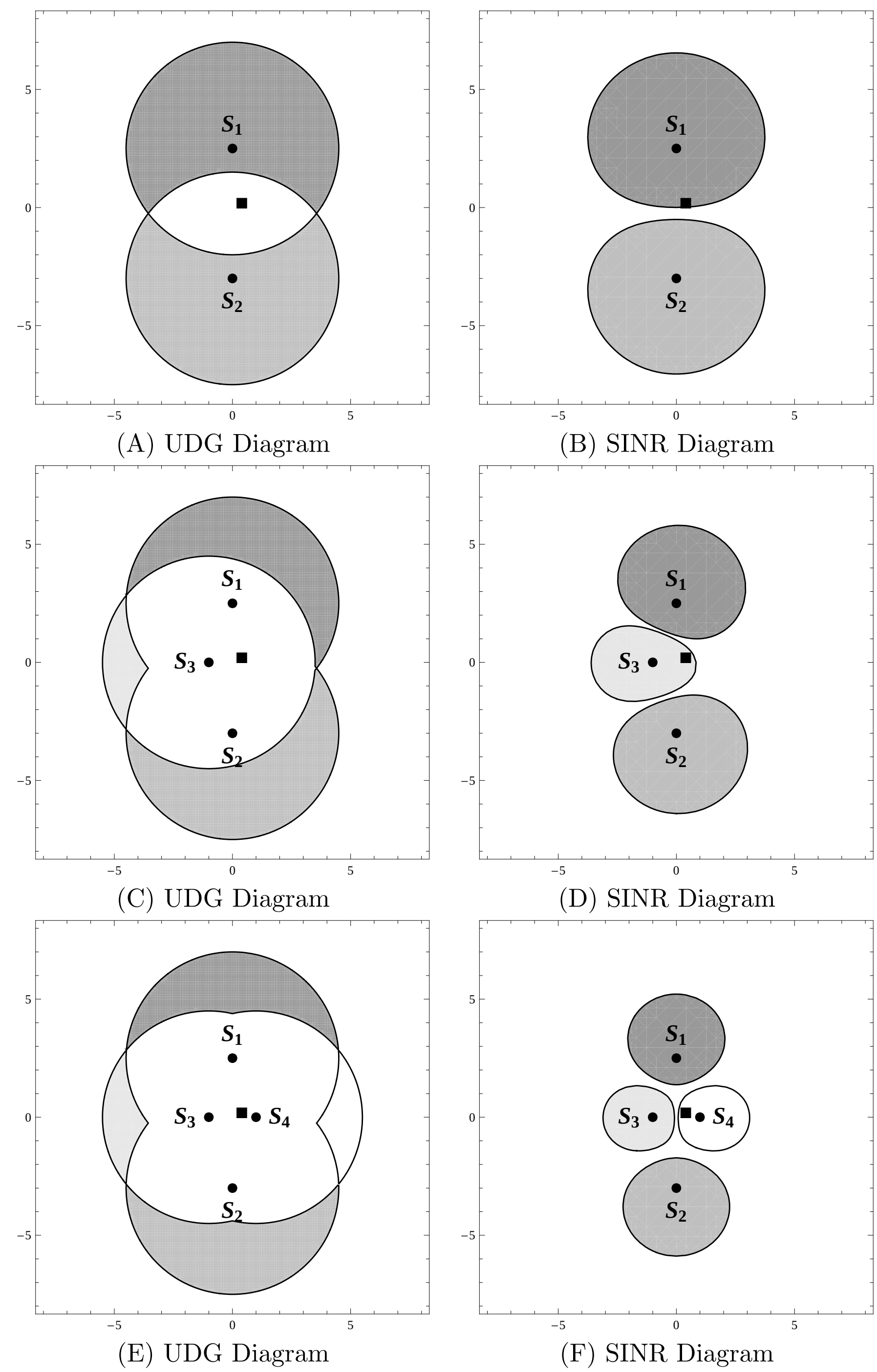


Figure 3: Reception diagrams in the UDG and SINR models. Initially only s_1 transmits, so the reception zone is the same in both models.

In summary, while the existing body of literature on models and algorithms for wireless networks



(A) UDG Diagram (B) SINR Diagram

(C) UDG Diagram (D) SINR Diagram

(E) UDG Diagram (F) SINR Diagram

Figure 4: Reception zones in the UDG and SINR models when adding stations one at a time. (A)-(B): adding s_2. (C)-(D): adding s_3. (E)-(F): adding s_4.

represents a significant base containing a rich collection of tools and techniques, the state of affairs described above leaves us in the unfortunate situation where the more practical graph-based models (such as the UDG model) are not sufficiently accurate, and the more accurate physical models are not well-understood and therefore difficult to use by protocol designers. Hence obtaining a better understanding of the SINR model, and consequently bridging the gap between this physical model and the graph based models, may have potentially significant (theoretical and practical) implications. This goal is the central motivation behind the current paper.

1.2 Related work

Some recent studies aim at achieving a better understanding of the SINR model. In particular, in their seminal work [12], Gupta and Kumar analyzed the capacity of wireless networks in the physical and protocol models. Moscibroda [17] analyzed the worst-case capacity of wireless networks, making no assumptions on the deployment of nodes in the plane, as opposed to almost all the previous work on wireless network capacity.

Thought provoking experimental results presented in [19] show that even basic wireless stations can achieve communication patterns that are impossible in graph-based models. Moreover, the paper presents certain situations in which it is possible to apply routing / transport schemes that may break the theoretical throughput limits of any protocol which obeys the laws of a graph-based model.

Moscibroda and Wattenhofer were the first to study non-trivial SINR effects in an algorithmic context, focusing on basic scheduling questions [18]. A related line of research, in which known results from the UDG model are analyzed under the SINR model, includes [20], which studies the problem of topology control in the SINR model, and [10], where impossibility results were proven in the SINR model for scheduling.

More elaborate graph-based models may employ two separate graphs, a connectivity graph $G_c = (S, E_c)$ and an interference graph $G_i = (S, E_i)$, such that a station s will successfully receive a message transmitted by a station s' if and only if s and s' are neighbors in the connectivity graph G_c and s does not have a concurrently transmitting neighbor in the interference graph G_i. Protocol designers often consider special cases of this more general model. For example, it is sometimes assumed that G_i is G_c augmented with all edges between 2-hop neighbors in G_c. Similarly, a variant of the UDG model handling transmissions and interference separately, named the *Quasi Unit Disk Graph (Q-UDG)* model, was introduced in [15]. In this model, two concentric circles are associated with each station, the smaller representing its reception zone and the larger representing its area of interference. An alternative interference model, also based on the UDG model, is proposed in [23].

A constant factor approximation algorithm for scheduling arbitrary sets of wireless links in uniform power networks was obtained in [11]. In[14] it was proven that wireless scheduling in R^2 with $\alpha > 2$ is in APX. An $O(\log n \log\log \Delta)$ approximation algorithm for SINR scheduling in the case of unidirectional links, where Δ is the ratio between the longest and the shortest link length, was presented in [13].

A natural question concerns the difference between the arbitrary and uniform power models. It is possible to prove that if some resource (e.g., the energy used or the general area in which the network resides) is bounded, then the ratio between the two models is proportional to $\log B$, where B is the bound on the resource; see [3, 4, 13, 16].

1.3 Our results

As mentioned earlier, a fundamental issue in wireless network modeling involves characterizing the reception zones of the stations and constructing the reception diagram. The current paper aims at gaining a better understanding of this issue in the SINR model, and as a consequence, deriving some algorithmic results. In particular, we consider the structure of reception zones in SINR diagrams corresponding to uniform power networks in a d-dimensional space ($d \in \mathbb{Z}_{\geq 1}$) with path-loss parameter $\alpha > 0$, and examine two specific properties of interest, namely, the *convexity* and *fatness* of the reception zones. (The notion of fatness has received a number of non-equivalent technical definitions, all aiming at capturing the same intuition, namely, absence of long, skinny or twisted parts. In this paper we say that the reception zone of station s_i is *fat* if the ratio between the radii of the smallest ball centered at s_i that completely contains the zone and the largest ball centered at s_i that is completely contained by it is bounded by some constant. Refer to Section 2.1 for a formal definition.) Apart from their theoretical interest, these properties are also of considerable practical significance, as obviously, having reception zones that are non-convex, or whose shape is arbitrarily skewed, twisted or skinny, might complicate the development of protocols for various design and communication tasks.

Our first result is cast in Theorem 1, proven in Section 3.

Theorem 1. *The reception zones in an SINR diagram of a uniform power network in a d-dimensional space with path-loss parameter $\alpha > 0$ and reception threshold $\beta \geq 1$ are convex.*

Note that our convexity proof still holds when $\beta = 1$. In contrast, when $\beta < 1$, the reception zones of a uniform power network are not necessarily convex, and may also overlap. This phenomenon is illustrated in (the numerically generated) Figure 5. We then establish an additional attractive property of the reception zones.

Theorem 2. *The reception zones in an SINR diagram of a uniform power network in a d-dimensional space with path-loss parameter $\alpha > 0$ and reception threshold $\beta > 1$ are fat.*

Theorem 2 is proved in Section 4. In a certain sense, this result lends support to the model of *Quasi Unit Disk Graphs* suggested by Kuhn et al. in [15].

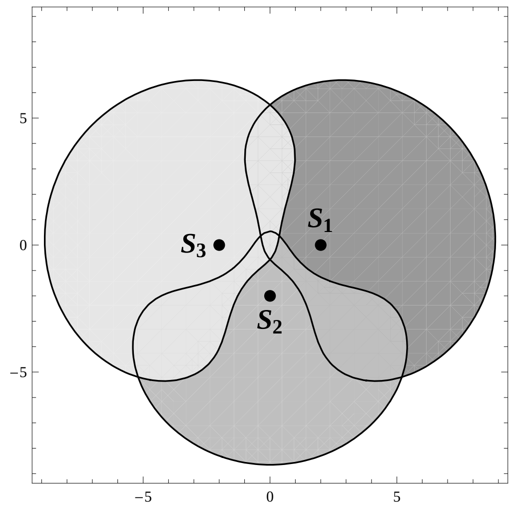


Figure 5: A uniform power network with three transmitters s_1, s_2, s_3, path-loss parameter $\alpha = 2$, reception threshold $\beta = 0.3$, and background noise $N = 0.05$. The reception zones (bold lines) are clearly non-convex and overlapping.

Armed with this characterization of the reception zones, we turn to a basic algorithmic task closely related to SINR diagrams, namely, answering *point location queries*. We address the following natural question: given a point in the plane, which reception zone contains this point (if any)? For UDG, this problem can be dealt with using known techniques, cf. [1, 2]. For arbitrary (non-unit) disk graphs, the problem is already harder, as the direct reduction to the technique of [2] no longer works. In the SINR model the problem becomes even harder. A naive solution will require computing the signal to interference & noise ratio for each station, yielding time $O(n^2)$. A more efficient ($O(n)$ time) querying algorithm can be based, for example, on the observation that there is a unique candidate $s_i \in S$ whose transmission may be received at p, that is, the one whose Voronoi cell contains p in the Voronoi diagram defined for S. However, it is not known if a sublinear query time can be obtained. This problem can in fact be thought of as part of a more general one, i.e., point location over a general set of objects satisfying some "niceness" properties. Previous work on the problem dealt with Tarski cells, namely, objects whose boundaries are defined by a constant number of polynomials of constant degree [1, 7]. In contrast, the SINR diagram consists of objects (the reception zones) whose boundaries may be defined by polynomials[2] of degree proportional to n. We are unaware of a technique that answers point location queries for such objects in sublinear time.

Consider the SINR diagram of a uniform power network in a 2-dimensional space (the Euclidean plane) with path-loss parameter $\alpha > 0$ and reception threshold $\beta > 1$ and fix some performance parameter $0 < \epsilon < 1$. The following theorem is proved in Section 5 (refer to Figure 6 for illustration).

Theorem 3. *For every n-station uniform power network with reception zones $\langle \mathcal{H}_0, \ldots, \mathcal{H}_{n-1} \rangle$, it*

[2] In fact, when α is not an integer, the boundaries of the reception zones are not defined by polynomials at all.

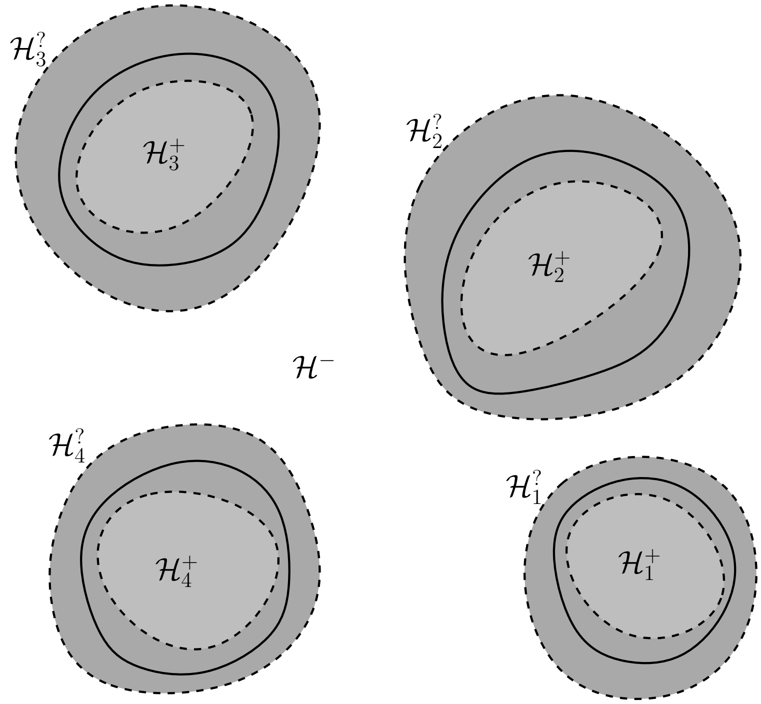


Figure 6: The reception zones $\mathcal{H}_i$ for $1 \le i \le 4$ (enclosed by bold lines) and the partition of the plane into disjoint zones $\mathcal{H}_i^+$ (light gray, enclosed by inner dashed lines), $\mathcal{H}_i^?$ (dark gray, between inner and outer dashed lines), and $\mathcal{H}^-$ (the remaining white).

is possible to construct, in $O(n^2\epsilon^{-1})$ preprocessing time, a data structure `DS` *requiring memory of size $O(n\epsilon^{-1})$, that imposes a $(2n+1)$-wise partition $\bar{\mathcal{H}} = \langle \mathcal{H}_0^+, \ldots, \mathcal{H}_{n-1}^+, \mathcal{H}_0^?, \ldots, \mathcal{H}_{n-1}^?, \mathcal{H}^- \rangle$ of the Euclidean plane $\mathbb{R}^2$ (that is, the zones in $\bar{\mathcal{H}}$ are pair-wise disjoint and $\mathbb{R}^2 = \bigcup_{i=0}^{n-1} \mathcal{H}_i^+ \cup \mathcal{H}^- \cup \bigcup_{i=0}^{n-1} \mathcal{H}_i^?$) such that for every $0 \le i \le n-1$:*

(1) $\mathcal{H}_i^+ \subseteq \mathcal{H}_i$;

(2) $\mathcal{H}^- \cap \mathcal{H}_i = \emptyset$; and

(3) $\mathcal{H}_i^?$ is bounded and its area is at most an ϵ-fraction of the area of $\mathcal{H}_i$.

Furthermore, given a query point $p \in \mathbb{R}^2$, it is possible to extract from `DS`*, in time $O(\log n)$, the zone in $\bar{\mathcal{H}}$ to which p belongs.*

1.4 Open Problems

Our results concern wireless networks with uniform power transmissions. General wireless networks are harder to deal with. For instance, the point location problem becomes considerably more difficult when different stations are allowed to use different transmission energy, since in this case, the appropriate graph-based model is no longer a unit-disk graph but a (*directed*) general disk graph, based on disks of arbitrary radii. An even more interesting case is the *variable power* setting, where the stations can adjust their transmission energy levels from time to time.

The problems discussed above become harder in a dynamic setting, and in particular, if we assume the stations are mobile, and extending our approach to the dynamic and mobile settings are the natural next steps.

2 Preliminaries

2.1 Geometric notions

Throughout, we consider the d-dimensional Euclidean space $\mathbb{R}^d$ (for $d \in \mathbb{Z}_{\geq 1}$). The *distance* between points p and point q is denoted by $\mathrm{dist}(p, q) = \|q - p\|$. We extend the definition of distance to point sets so that the distance between point sets P and Q is $\mathrm{dist}(P, Q) = \inf\{\mathrm{dist}(p, q) \mid p \in P, q \in Q\}$. A *ball* of radius r centered at point p is the set of all points at distance at most r from p, denoted by $B(p, r) = \{q \in \mathbb{R}^d \mid \mathrm{dist}(p, q) \leq r\}$. We say that point $p \in \mathbb{R}^d$ is *internal* to the point set P if there exists some $\epsilon > 0$ such that $B(p, \epsilon) \subseteq P$.

A point set P is said to be *open* if all points $p \in P$ are internal points, and *closed* if its complement $\bar{P}$ is open. If there exists some real r such that $\mathrm{dist}(p, q) \leq r$ for every two points $p, q \in P$, then P is said to be *bounded*. A *compact* set is a set that is both closed and bounded. The *closure* of P, denoted $cl(P)$, is the smallest closed set containing P. The *boundary* of P, denoted by ∂P, is the intersection of the closure of P and the closure of its complement, i.e., $\partial P = cl(P) \cap cl(\bar{P})$. A *connected set* is a point set P that cannot be partitioned into two non-empty subsets P_1, P_2 such that each of the subsets has no point in common with the closure of the other, i.e., P is connected if for every $P_1, P_2 \neq \emptyset$ such that $P_1 \cap P_2 = \emptyset$ and $P_1 \cup P_2 = P$, either $P_1 \cap cl(P_2) \neq \emptyset$ or $P_2 \cap cl(P_1) \neq \emptyset$. We refer to the closure of an open bounded connected set as a *thick* set. By definition, every thick set is compact. We now turn to define the notion of convexity which plays a major role in this paper.

Definition 2.1. A point set P is said to be *convex* if the segment $\overline{pq}$ is contained in P for every two points $p, q \in P$.

The point set P is said to be *star-shaped* [9] with respect to point $p \in P$ if the segment $\overline{pq}$ is contained in P for every point $q \in P$. Clearly, convexity is stronger than the star-shape property in the sense that a convex point set P is star-shaped with respect to any point $p \in P$; the converse is not necessarily true in the sense that the star-shape property with respect to a single point does not imply convexity.

We frequently use the term *zone* to describe a point set with some "niceness" properties. Unless stated otherwise, a zone refers to the union of an open connected set and some subset of

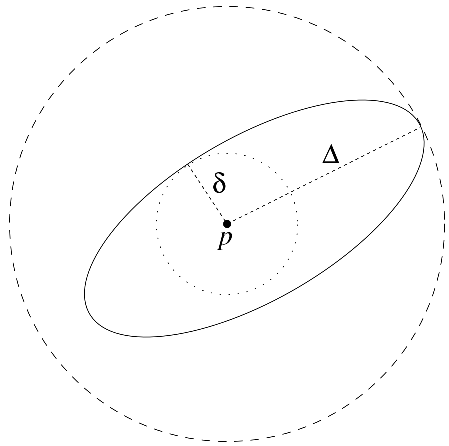


Figure 7: The zone Z (enclosed by the solid line) with the ball defining $\delta(p, Z)$ (dotted line) and the ball defining $\Delta(p, Z)$ (dashed line).

its boundary. (A thick set is a special case of a zone.) It may also refer to a single point or to the finite union of zones. Given some bounded zone Z, we denote the *area* of Z (assuming that it is well-defined) by area(Z). For a non-empty bounded zone Z and an internal point p of Z, denote

$$\delta(p, Z) = \sup\{r > 0 \mid Z \supseteq B(p, r)\} , \qquad \Delta(p, Z) = \inf\{r > 0 \mid Z \subseteq B(p, r)\} ,$$

and define the *fatness parameter* of Z *with respect to* p to be $\varphi(p, Z) = \Delta(p, Z)/\delta(p, Z)$. (See Figure 7.) The zone Z is said to be *fat* with respect to p if $\varphi(p, Z)$ is bounded by some constant.

2.2 Wireless networks

We consider a wireless network $\mathcal{A} = \langle d, S, \psi, N, \beta, \alpha \rangle$, where $d \in \mathbb{Z}_{\geq 1}$ is the dimension, $S = \{s_0, s_1, \ldots, s_{n-1}\}$ is a set of transmitting *radio stations* embedded in the d-dimensional space, ψ is an assignment of a positive real *transmitting power* ψ_i to each station s_i, $N \geq 0$ is the *background noise*, $\beta \geq 1$ is a constant that serves as the *reception threshold* (to be explained soon), and $\alpha > 0$ is the *path-loss parameter*. For notational simplicity, s_i also refers to the point $(x_1^i, ..., x_d^i)$ in the d-dimensional space $\mathbb{R}^d$ where the station s_i resides, and moreover, when $d = 2$, the point s_i in the Euclidean plane is denoted (a_i, b_i). The network is assumed to contain at least two stations, i.e., $n \geq 2$. (In general, the station locations are not assumed to be distinct; whenever distinctness is required, we state it explicitly.) We say that $\mathcal{A}$ is a *uniform power network* if $\psi = \bar{1}$, namely, $\psi_i = 1$ for every $0 \leq i \leq n-1$.

The *energy* of station s_i at point $p \neq s_i$ is defined to be $\mathrm{E}_{\mathcal{A}}(s_i, p) = \psi_i \cdot \mathrm{dist}(s_i, p)^{-\alpha}$. The *energy* of a set of stations $T \subseteq S$ at a point $p \notin T$ is defined to be $\mathrm{E}_{\mathcal{A}}(T, p) = \sum_{s_i \in T} \mathrm{E}_{\mathcal{A}}(s_i, p)$. Fix some station s_i and consider some point $p \notin S$. We define the *interference* to s_i at point p to be the energies at p of all stations other than s_i, denoted $\mathrm{I}_{\mathcal{A}}(s_i, p) = \mathrm{E}_{\mathcal{A}}(S - \{s_i\}, p)$. The *signal to* *interference & noise ratio (SINR)* of s_i at point p is defined as

$$\mathrm{SINR}_{\mathcal{A}}(s_i, p) \;=\; \frac{\mathrm{E}_{\mathcal{A}}(s_i, p)}{\mathrm{I}_{\mathcal{A}}(s_i, p) + N} \;=\; \frac{\psi_i \cdot \mathrm{dist}(s_i, p)^{-\alpha}}{\sum_{j \neq i} \psi_j \cdot \mathrm{dist}(s_j, p)^{-\alpha} + N} \;. \tag{1}$$

Observe that $\mathrm{SINR}_{\mathcal{A}}(s_i, p)$ is always positive since the transmitting powers and the distances of the stations from p are always positive and the background noise is non-negative. When the network $\mathcal{A}$ is clear from the context, we may omit it and write simply $\mathrm{E}(s_i, p)$, $\mathrm{I}(s_i, p)$, and $\mathrm{SINR}(s_i, p)$.

The fundamental rule of the SINR model is that the transmission of station s_i is received correctly at point $p \notin S$ if and only if its SINR at p is not smaller than the reception threshold of the network, i.e., $\mathrm{SINR}(s_i, p) \geq \beta$. If this is the case, then we say that s_i is *heard* at p. We refer to the set of points that hear station s_i as the *reception zone* of s_i, defined as

$$\mathcal{H}_i = \{p \in \mathbb{R}^d - S \mid \mathrm{SINR}(s_i, p) \geq \beta\} \cup \{s_i\} \;.$$

This admittedly tedious definition is necessary as $\mathrm{SINR}(s_i, \cdot)$ is not defined at any point in S and in particular, at s_i itself.

Certain relationships hold between the SINR diagram on a set of stations S and the corresponding *Voronoi diagram* on S, cf. [9]. The Voronoi diagram partitions[3] the d-dimensional space into n *Voronoi cells*, denoted $\textsc{Vor}(s_i)$ for $s_i \in S$, such that $\textsc{Vor}(s_i) = \{p \in \mathbb{R}^2 \mid \mathrm{dist}(s_i, p) < \mathrm{dist}(s_j, p)$ for any $j \neq i\}$.

A uniform power network $\mathcal{A} = \langle d, S, \bar{1}, N, \beta, \alpha \rangle$ is said to be *trivial* if $|S| = 2$, $N = 0$, and $\beta = 1$. Note that for $i = 0, 1$, the reception zone $\mathcal{H}_i$ of station s_i in a trivial uniform power network is the half-plane consisting of all points whose distance to s_i is not greater than their distance to s_{1-i}. In particular, $\mathcal{H}_i$ is unbounded. Hence for trivial networks, the reception zones of the SINR diagram coincide with the closure of the Voronoi cells.

For non-trivial networks, the SINR diagram no longer partitions the space, and its n reception zones are strictly contained in the corresponding Voronoi cells. This is expressed formally in the following observation, which relies on the fact that $\mathrm{SINR}(s_i, \cdot)$ is a continuous function in $\mathbb{R}^d - S$.

Observation 2.2. *Let $\mathcal{A} = \langle d, S, \bar{1}, N, \beta, \alpha \rangle$ be a non-trivial uniform power network (i.e., with $n \geq 3$ or $N > 0$ or $\beta > 1$). Then the reception zone $\mathcal{H}_i$ is compact for every $s_i \in S$. Moreover, every point in $\mathcal{H}_i$ is closer to s_i than it is to any other station in S, i.e., $\mathcal{H}_i$ is strictly contained in the Voronoi cell* $\textsc{Vor}(s_i)$*.*

For a nontrivial uniform power network $\mathcal{A}$, we say that the reception zone $\mathcal{H}_i$ is *fat* if it is fat with respect to s_i.

Next, we state a simple but important lemma that will be useful in our later arguments.

[3] Strictly speaking, the Voronoi diagram partitions the space into n Voronoi cells and the remaining points residing on the boundaries of these cells.

Lemma 2.3. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a mapping consisting of rotation, translation, and scaling by a factor of $\sigma > 0$. Consider some network $\mathcal{A} = \langle d, S, \psi, N, \beta, \alpha \rangle$ and let $f(\mathcal{A}) = \langle d, f(S), \psi, N/\sigma^\alpha, \beta, \alpha \rangle$, where $f(S) = \{f(s_i) \mid s_i \in S\}$. Then for every station s_i and for all points $p \notin S$, we have* $\mathrm{SINR}_{\mathcal{A}}(s_i, p) = \mathrm{SINR}_{f(\mathcal{A})}(f(s_i), f(p))$.

Proof. Consider a mapping $f : \mathbb{R}^d \to \mathbb{R}^d$ consisting of rotation, translation, and scaling by some constant factor $\sigma > 0$. It is clear that, $\mathrm{dist}(f(p), f(q)) = \sigma \cdot \mathrm{dist}(p, q)$, for every two points $p, q \in \mathbb{R}^d$. Therefore,

$$\mathrm{E}_{f(\mathcal{A})}(s_i, p) = \frac{\psi_i}{(\sigma \cdot \mathrm{dist}(s_i, p))^\alpha} = \frac{\mathrm{E}_{\mathcal{A}}(s_i, p)}{\sigma^\alpha}$$

and

$$\mathrm{I}_{f(\mathcal{A})}(s_i, p) = \sum_{j \neq i} \mathrm{E}_{f(\mathcal{A})}(s_j, p) = \sum_{j \neq i} \frac{\mathrm{E}_{\mathcal{A}}(s_j, p)}{\sigma^\alpha} = \frac{\mathrm{I}_{\mathcal{A}}(s_i, p)}{\sigma^\alpha}.$$

Combining these equalities with Eq. (1), we have

$$\begin{aligned} \mathrm{SINR}_{f(\mathcal{A})}(f(s_i), f(p)) &= \frac{\mathrm{E}_{f(\mathcal{A})}(s_i, p)}{\mathrm{I}_{f(\mathcal{A})}(s_i, p) + N/\sigma^\alpha} \\ &= \frac{\mathrm{E}_{\mathcal{A}}(s_i, p)/\sigma^\alpha}{\mathrm{I}_{\mathcal{A}}(s_i, p)/\sigma^\alpha + N/\sigma^\alpha} = \mathrm{SINR}_{\mathcal{A}}(s_i, p). \end{aligned}$$

The lemma follows. □

2.3 Technical claims

For completeness, the proofs of the following technical lemma and observation are included in the appendix.

Lemma 2.4. *For all $x, y, z \in (0, 1)$ and for all $\sigma, \tau, \alpha \in \mathbb{R}_{>0}$, we have*

$$\max\left\{ \sigma \left(\frac{x}{y}\right)^\alpha + \tau \left(\frac{x}{z}\right)^\alpha, \sigma \left(\frac{1-x}{1-y}\right)^\alpha + \tau \left(\frac{1-x}{1-z}\right)^\alpha \right\} \ \geq \ \sigma + \tau \ .$$

Observation 2.5. $\frac{\sqrt[\alpha]{a+c}+1}{\sqrt[\alpha]{b+c}-1} \leq \frac{\sqrt[\alpha]{a}+1}{\sqrt[\alpha]{b}-1}$ *for any reals $a \geq b > 1$, $c > 0$, and $\alpha > 0$.*

3 Convexity of the reception zones

In this section we consider the SINR diagram of a uniform power network $\mathcal{A} = \langle d, S, \bar{1}, N, \beta, \alpha \rangle$ and establish Theorem 1. As all stations admit the same transmitting power, it is sufficient to focus on s_0 and to prove that the reception zone $\mathcal{H}_0$ is convex. We begin by proving this assertion on the Euclidean plane, i.e., for dimension $d = 2$. We do so by considering some arbitrary two points $p_1, p_2 \in \mathbb{R}^2$ and arguing that if s_0 is heard at both p_1 and p_2, then s_0 is heard at all points in the segment $\overline{p_1 \, p_2}$. This argument is established in three steps. First, as a warmup, we prove that $\mathcal{H}_0$

is star-shaped with respect to s_0. This proof, presented in Section 3.1, establishes our argument for the case that p_1 and p_2 are collinear with s_0. It also implies the correctness of Theorem 1 for the case $d = 1$. Second, we prove that in the absence of background noise (i.e., $N = 0$), if $p_i \in \mathcal{H}_0$ for $i = 1, 2$, then $\overline{p_1 p_2} \subseteq \mathcal{H}_0$. This proof, presented in Section 3.3, relies on the analysis of a special case of a network consisting of only three stations which is analyzed in Section 3.2. Third, in Section 3.4 we reduce the convexity proof of a uniform power network with n stations and arbitrary background noise to that of a uniform power network with $n+1$ stations and no background noise, thus completing the proof for the 2-dimensional case. Finally, in Section 3.5 we prove this assertion on a d-dimensional space for any integer $d \geq 2$.

3.1 Star-shape for the 2-dimensional case

In this section we consider a uniform power network $\mathcal{A} = \langle 2, S, \bar{1}, N, \beta, \alpha \rangle$, and show that the reception zone $\mathcal{H}_0$ is star-shaped with respect to the station s_0. In fact, we prove a slightly stronger lemma.

Lemma 3.1. *Consider some point* $p \in \mathbb{R}^2$. *If* $\mathrm{SINR}(s_0, p) \geq 1$, *then* $\mathrm{SINR}(s_0, q) > \mathrm{SINR}(s_0, p)$ *for all internal points* q *in the segment* $\overline{s_0 p}$.

Proof. We consider two disjoint cases. First, suppose that there exists some station s_i, $i > 0$, such that $\mathrm{E}(s_i, p) = \mathrm{E}(s_0, p)$. The assumption that $\mathrm{SINR}(s_0, p) \geq 1$ necessitates, by (1), that $N = 0$, $n = 2$ (which means that $i = 1$), and $\mathrm{SINR}(s_0, p) = 1$. Therefore $\mathrm{dist}(s_0, p) = \mathrm{dist}(s_1, p)$ and for all internal points q in the segment $\overline{s_0 p}$ we have $\mathrm{dist}(s_0, q) < \mathrm{dist}(s_1, q)$. Thus $\mathrm{SINR}(s_0, q) > 1$ and the assertion holds.

Now, suppose that $\mathrm{E}(s_i, p) < \mathrm{E}(s_0, p)$ for every $i > 0$, which means that $\mathrm{dist}(s_i, p) > \mathrm{dist}(s_0, p)$ for every $i > 0$. By Lemma 2.3, we may assume without loss of generality that $s_0 = (0, 0)$ and $p = (-1, 0)$. Consider some station s_i, $i > 0$. Note that if s_i is not located on the positive half of the horizontal axis, then it can be relocated to a new location s'_i on the positive half of the horizontal axis by rotating it around p so that $\mathrm{dist}(s'_i, p) = \mathrm{dist}(s_i, p)$ and $\mathrm{dist}(s'_i, q) \leq \mathrm{dist}(s_i, q)$ for all points $q \in \overline{s_0 p}$ (see Figure 8). This process can be repeated with every station s_i, $i > 0$, until all stations are located on the positive half of the horizontal axis without decreasing the interference at any point $q \in \overline{s_0 p}$. Therefore it is sufficient to establish the assertion under the assumption that $s_i = (a_i, 0)$, where $a_i > 0$, for every $i > 0$.

Let $q = (-x, 0)$ for some $x \in (0, 1]$. The SINR function of s_0 at q can be expressed as

$$\mathrm{SINR}(s_0, q) \;=\; \frac{x^{-\alpha}}{\sum_{i>0} (a_i + x)^{-\alpha} + N} \;.$$

In this context, it will be more convenient to consider the reciprocal of the SINR function,

$$f(x) \;=\; \mathrm{SINR}^{-1}(s_0, q) \;=\; \sum_{i>0} \left(\frac{x}{a_i + x} \right)^{\alpha} + x^{\alpha} \cdot N \;,$$

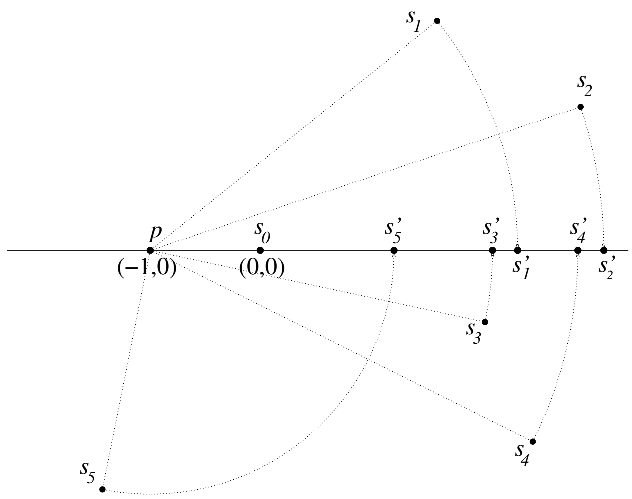


Figure 8: Relocating stations s_i, $i > 0$.

so that it remains to prove that $f(x) < f(1)$ for all $x \in (0,1)$. The assertion follows since the derivative $\frac{df(x)}{dx} = \alpha x \cdot \sum_{i>0} \frac{a_i}{(a_i+x)^{(\alpha+1)}} + \alpha x \cdot N$ is positive when $x \in (0,1]$. □

Consider a non-trivial uniform power network $\mathcal{A} = \langle 2, S, \bar{1}, N, \beta, \alpha \rangle$ and suppose that $s_0 \neq s_j$ for every $j > 0$, that is, the location of s_0 is not shared by any other station. Lemma 3.1 implies that the point set $\mathcal{H}'_0 = \{p \in \mathbb{R}^2 - S \mid \text{SINR}_{\mathcal{A}}(s_0, p) > \beta\} \cup \{s_0\}$ is star-shaped with respect to s_0, and in particular, connected. Moreover, since SINR is a continuous function in $\mathbb{R}^2 - S$, it follows that $\mathcal{H}'_0$ is an open set. As $\mathcal{H}_0$ is the closure of $\mathcal{H}'_0$, we have the following corollary.

Corollary 3.2. *In a non-trivial network, if the location of s_0 is not shared by any other station, then $\mathcal{H}_0$ is a thick set. Moreover, it is star-shaped with respect to s_0.*

3.2 Convexity for three stations with no background noise

In this section we analyze the special case of the 3-station uniform power network $\mathcal{A}_3 = \langle 2, S, \bar{1}, N, \beta, 2\alpha \rangle$, where $S = \{s_0, s_1, s_2\}$, $N = 0$, $\beta = 1$ and[4] $2\alpha > 0$. Recall that for this special case, the SINR and SINR^{-1} formulas take the form

$$\text{SINR}_{\mathcal{A}}(s_0, p) \;=\; \frac{\frac{1}{\text{dist}(s_0,p)^{2\alpha}}}{\frac{1}{\text{dist}(s_1,p)^{2\alpha}} + \frac{1}{\text{dist}(s_2,p)^{2\alpha}}}$$

and

$$\text{SINR}^{-1}_{\mathcal{A}}(s_0, p) \;=\; \left(\frac{\text{dist}(s_0,p)^2}{\text{dist}(s_1,p)^2}\right)^{\alpha} + \left(\frac{\text{dist}(s_0,p)^2}{\text{dist}(s_2,p)^2}\right)^{\alpha}. \qquad (2)$$

Our goal is to establish the following lemma.

[4] The path-loss parameter is denoted here as 2α to simplify the analysis.

Lemma 3.3. *The reception zone $\mathcal{H}_0$ of station s_0 in $\mathcal{A}_3$ is convex.*

In order to establish Lemma 3.3, it is required to show that $\overline{p_1 p_2} \subseteq \mathcal{H}_0$ for any two points $p_1, p_2 \in \mathcal{H}_0$, i.e., $\text{SINR}_{\mathcal{A}}(s_0, q) \geq \beta$ for any $q \in \overline{p_1 p_2}$. By Lemma 3.1, $\overline{p_1 p_2} \subseteq \mathcal{H}_0$ for any $p_1, p_2 \in \mathcal{H}_0$ which are collinear with s_0. Here we prove the claim for the remaining cases. In fact, we prove a stronger claim, namely, that

$$\begin{aligned}&\text{SINR}_{\mathcal{A}}(s_0, q) \geq \min\{\text{SINR}_{\mathcal{A}}(s_0, p_1), \text{SINR}_{\mathcal{A}}(s_0, p_2)\} \geq \beta \\ &\text{for any } p_1, p_2 \in \mathcal{H}_0 \text{ such that } s_0 \notin \overline{p_1 p_2} \text{ and } q \in \overline{p_1 p_2}\ .\end{aligned}$$

Actually, the claim we prove is even slightly more general than that, as it shows that the SINR function is convex in any segment that is contained in $\text{Vor}(s_0)$, the Voronoi cell of s_0 (recall that $\mathcal{H}_0 \subseteq \text{Vor}(s_0)$). Formally, we show that

$$\begin{aligned}&\text{SINR}_{\mathcal{A}}(s_0, q) \geq \min\{\text{SINR}_{\mathcal{A}}(s_0, p_1), \text{SINR}_{\mathcal{A}}(s_0, p_2)\} \\ &\text{for any } p_1, p_2 \in \text{Vor}(s_0) \text{ such that } s_0 \notin \overline{p_1 p_2} \text{ and } q \in \overline{p_1 p_2}\ .\end{aligned}$$

The SINR function is continuous in both variables (x and y) and in particular on any straight line or segment (except on the points s_0, s_1 and s_2). Therefore, in order to prove the convexity of SINR in $\text{Vor}(s_0)$, it is sufficient to prove the following. For any two points $p_1, p_2 \in \mathbb{R}^2$, let q denote the middle point of the segment $\overline{p_1 p_2}$, i.e., $q \in \overline{p_1 p_2}$ and $\text{dist}(p_1, q) = \text{dist}(p_2, q)$. Then we show that

$$\begin{aligned}&\text{SINR}_{\mathcal{A}}(s_0, q) \geq \min\{\text{SINR}_{\mathcal{A}}(s_0, p_1), \text{SINR}_{\mathcal{A}}(s_0, p_2)\} \\ &\text{for any } p_1, p_2 \in \mathbb{R}^2 \text{ such that } s_0, s_1, s_2 \notin \overline{p_1 p_2} \text{ and } q \in \text{Vor}(s_0)\ .\end{aligned} \tag{3}$$

To prove Eq. (3), consider points p_1, p_2, q as above. By Lemma 2.3 we may assume without loss of generality that $p_1 = (-1, 0)$, $p_2 = (1, 0)$ and $q = (0, 0)$. Consider the uniform power network $\mathcal{A}' = \langle S' = \{s'_0, s'_1, s'_2\}, \bar{1}, N = 0, \beta = 1, 2\alpha \rangle$ obtained from $\mathcal{A}$ by rotating each of the stations s_0, s_1 and s_2 (separately) around the origin point q until it reaches the positive y-axis, i.e., the stations s'_0, s'_1 and s'_2 are on the positive y-axis and preserve the distances of s_0, s_1 and s_2, respectively, from q, and hence $\text{SINR}_{\mathcal{A}}(s_0, q) = \text{SINR}_{\mathcal{A}'}(s'_0, q)$. Formally, letting $\rho_i = ||s_i|| = \sqrt{a_i^2 + b_i^2}$, for $i = 0, 1, 2$, the station s'_i is located at $(0, \rho_i)$ for $i = 0, 1, 2$, as illustrated in Figure 9. As $q \in \text{Vor}(s_0)$, we have:

Observation 3.4. *$\rho_i > \rho_0$ for $i = 1, 2$.*

We are now ready to establish Lemma 3.3. This is done by combining the following two propositions, thus proving the validity of Eq. 3.

Proposition. $\text{SINR}_{\mathcal{A}}(s_0, q) > \text{SINR}_{\mathcal{A}'}(s'_0, p_1) = \text{SINR}_{\mathcal{A}'}(s'_0, p_2)$.

Proof. By Obs. 3.4,

$$\frac{\rho_0^2}{\rho_i^2} = \frac{\rho_0^2(\rho_i^2 + 1)}{\rho_i^2(\rho_i^2 + 1)} = \frac{\rho_0^2 + \rho_0^2/\rho_i^2}{\rho_i^2 + 1} < \frac{\rho_0^2 + 1}{\rho_i^2 + 1}$$

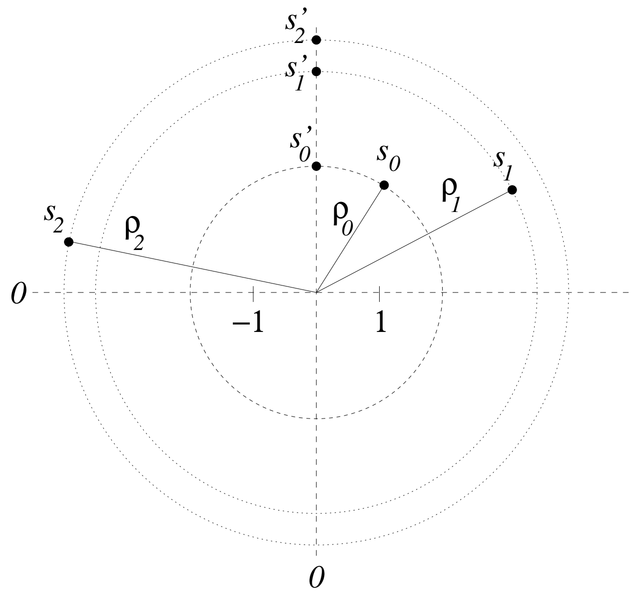


Figure 9: The network $\mathcal{A}$ contains the stations s_0, s_1 and s_2. The network $\mathcal{A}'$ contains the rotated stations s_0', s_1' and s_2'. Note that s_i and s_i' are at equal distance ρ_i from the origin and s_i' is on the positive y-axis, and thus $\mathrm{dist}(s_i', p_1) = \mathrm{dist}(s_i', p_2)$, for $i = 0, 1, 2$.

for $i = 1, 2$. Therefore,

$$\begin{aligned}\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, q) &= \left(\frac{\rho_0^2}{\rho_1^2}\right)^{\alpha} + \left(\frac{\rho_0^2}{\rho_2^2}\right)^{\alpha} \\ &< \left(\frac{\rho_0^2+1}{\rho_1^2+1}\right)^{\alpha} + \left(\frac{\rho_0^2+1}{\rho_2^2+1}\right)^{\alpha} = \mathrm{SINR}_{\mathcal{A}'}^{-1}(s_0', p_1) = \mathrm{SINR}_{\mathcal{A}'}^{-1}(s_0', p_2)\ .\end{aligned}$$

The assertion follows. □

Proposition. $\mathrm{SINR}_{\mathcal{A}'}(s_0', p_1) = \mathrm{SINR}_{\mathcal{A}'}(s_0', p_2) \geq \min\{\mathrm{SINR}_{\mathcal{A}}(s_0, p_1), \mathrm{SINR}_{\mathcal{A}}(s_0, p_2)\}$.

Proof. Define the angle θ_i so that $s_i = (\rho_i \cos\theta_i, \rho_i \sin\theta_i)$ for $i = 0, 1, 2$. We have

$$\begin{aligned}\mathrm{dist}(s_i, p_1)^2 &= \rho_i^2 \sin^2\theta_i + (\rho_i \cos\theta_i + 1)^2 = \rho_i^2 \sin^2\theta_i + \rho_i^2 \cos^2\theta_i + 2\rho_i \cos\theta_i + 1 \\ &= \rho_i^2 + 2\rho_i \cos\theta_i + 1,\end{aligned}$$

and analogously

$$\mathrm{dist}(s_i, p_2)^2 = \rho_i^2 - 2\rho_i \cos\theta_i + 1\ .$$

Thus, for $i = 0, 1, 2$,

$$\mathrm{dist}(s_i, p_1)^2 + \mathrm{dist}(s_i, p_2)^2 = 2(\rho_i^2 + 1)\ .$$

Let $x_i = \frac{\text{dist}(s_i,p_1)^2}{2(\rho_i^2+1)}$, for $i = 0, 1, 2$. It follows that $x_0, x_1, x_2 \in (0, 1)$, and that $1 - x_i = \frac{\text{dist}(s_i,p_2)^2}{2(\rho_i^2+1)}$ for $i = 0, 1, 2$. Let $a = \frac{\rho_0^2+1}{\rho_1^2+1}$ and $b = \frac{\rho_0^2+1}{\rho_2^2+1}$. Formula (2) implies that

$$\text{SINR}_{\mathcal{A}}^{-1}(s_0, p_1) = a^\alpha \left(\frac{x_0}{x_1}\right)^\alpha + b^\alpha \left(\frac{x_0}{x_2}\right)^\alpha$$

and

$$\text{SINR}_{\mathcal{A}}^{-1}(s_0, p_2) = a^\alpha \left(\frac{1 - x_0}{1 - x_1}\right)^\alpha + b^\alpha \left(\frac{1 - x_0}{1 - x_2}\right)^\alpha .$$

Recall that the angles of the corresponding stations s_0', s_1' and s_2' are $\theta_0' = \theta_1' = \theta_2' = \pi/2$, hence $\text{SINR}_{\mathcal{A}'}^{-1}(s_0', p_1) = \text{SINR}_{\mathcal{A}'}^{-1}(s_0', p_2) = a^\alpha + b^\alpha$. Applying Lemma 2.4 with $\tau = a^\alpha$, $\gamma = b^\alpha$, $x = x_0$, $y = x_1$ and $z = x_2$, we have that

$$\max\left\{a^\alpha \left(\frac{x_0}{x_1}\right)^\alpha + b^\alpha \left(\frac{x_0}{x_2}\right)^\alpha, a^\alpha \left(\frac{1 - x_0}{1 - x_1}\right)^\alpha + b^\alpha \left(\frac{1 - x_0}{1 - x_2}\right)^\alpha\right\} \geq a^\alpha + b^\alpha$$

for all $x_0, x_1, x_2 \in (0, 1)$. This, in turn, implies that

$$\max\left\{\text{SINR}_{\mathcal{A}}^{-1}(s_0, p_1), \text{SINR}_{\mathcal{A}}^{-1}(s_0, p_2)\right\} \geq \text{SINR}_{\mathcal{A}'}^{-1}(s_0', p_1) = \text{SINR}_{\mathcal{A}'}^{-1}(s_0', p_2) ,$$

yielding the assertion. □

3.3 Convexity for n stations with no background noise

In this section we return to a uniform power network $\mathcal{A} = \langle 2, S, \bar{1}, N, \beta, \alpha\rangle$ with an arbitrary number of stations, an arbitrary reception threshold $\beta \geq 1$, and $\alpha > 0$, but still on the Euclidean plane and with no background noise (i.e., $d = 2$ and $N = 0$). Our goal is to establish the convexity of $\mathcal{H}_0$.

Lemma 3.5. *For an n-station uniform power network $\mathcal{A} = \langle d = 2, S, \bar{1}, 0, \beta, \alpha\rangle$ the reception zone $\mathcal{H}_0$ of station s_0 in $\mathcal{A}$ is convex.*

Lemma 3.5 is proved by induction on the number of stations $n = |S|$. For the base of the induction, $n = 2$, note that the theorem clearly holds if s_0 and s_1 share the same location, as this implies that $\mathcal{H}_0 = \{s_0\}$. Furthermore, if $\beta = 1$, which means that $\mathcal{A}$ is trivial, then $\mathcal{H}_0$ is a half-plane, and in particular convex. So in what follows we assume that $s_0 \neq s_1$ and that $\beta > 1$.

The inductive step of the proof of Lemma 3.5 is more involved. We consider some arbitrary two points $p_1, p_2 \in \mathcal{H}_0$ and prove that $\overline{p_1\, p_2} \subseteq \mathcal{H}_0$, so by Definition 2.1 $\mathcal{H}_0$ is convex. Informally, we show that if there exist at least two stations other than s_0, then it is possible to discard one station and relocate the rest so that the interference at p_i remains unchanged for $i = 1, 2$ and the interference at q does not decrease for all points $q \in \overline{p_1\, p_2}$. By the inductive hypothesis, the segment $\overline{p_1\, p_2}$ is contained in $\mathcal{H}_0$ in the new setting, hence it is also contained in $\mathcal{H}_0$ in the original setting. This idea relies on the following lemma.

Lemma 3.6. *Consider the stations s_0, s_1, s_2 and some distinct two points $p_1, p_2 \in \mathbb{R}^2$. If $\mathrm{E}(s_0, p_i) \geq \mathrm{E}(\{s_1, s_2\}, p_i)$ for $i = 1, 2$, then there exists a point $s^* \in \mathbb{R}^2$ such that a new station placed at s^* satisfies*
(1) $\mathrm{E}(s^*, p_i) = \mathrm{E}(\{s_1, s_2\}, p_i)$ for $i = 1, 2$; and
(2) $\mathrm{E}(s^*, q) \geq \mathrm{E}(\{s_1, s_2\}, q)$, for all points q in the segment $\overline{p_1\, p_2}$.

Proof of Lemma 3.6. Let $\rho_i = 1/\sqrt[\alpha]{\mathrm{E}(\{s_1, s_2\}, p_i)}$ and let B_i be a ball of radius ρ_i centered at p_i for $i = 1, 2$. It is easy to verify that B_i consists of all points s such that placing a new station at s yields $\mathrm{E}(s, p_i) \geq \mathrm{E}(\{s_1, s_2\}, p_i)$. Assume without loss of generality that $\rho_1 \geq \rho_2$.

Proposition. *The circles ∂B_1 and ∂B_2 intersect.*

Proof. By Lemma 2.3, we may assume that $p_1 = (0, 0)$ and $p_2 = (c, 0)$ for some positive c. Since s_0 must be in both B_1 and B_2, it follows that the two balls cannot be disjoint. We establish the claim by showing that B_2 is not contained in B_1. Let us define a new uniform power network $\mathcal{A}'$ consisting of the stations s_1, s_2, and $s' = (c + \rho_2, 0)$ with no background noise. The points p_1 and p_2 are collinear with the station s', hence Lemma 3.1 may be employed to conclude that

$$\mathrm{SINR}_{\mathcal{A}'}(s', p_1) < \mathrm{SINR}_{\mathcal{A}'}(s', p_2)\ . \tag{4}$$

The construction of $\mathcal{A}'$ guarantees that $\mathrm{SINR}_{\mathcal{A}'}(s', p_2) = \mathrm{E}(s', p_2)/\mathrm{E}(\{s_1, s_2\}, p_2) = 1$. On the other hand, if $B_2 \subseteq B_1$, then s' is in B_1 (see Figure 10(A)), and thus $\mathrm{E}(s', p_1) \geq \mathrm{E}(\{s_1, s_2\}, p_1)$, which means that $\mathrm{SINR}_{\mathcal{A}'}(s', p_1) \geq 1$, in contradiction to inequality (4). Therefore ∂B_1 and ∂B_2 must intersect. The proposition holds. □

Let s^* be an intersection point of ∂B_1 and ∂B_2 (see Figure 10(B)). We now show that s^* satisfies the assertions of Lemma 3.6. Note that $\mathrm{E}(s, p_i) = \mathrm{E}(\{s_1, s_2\}, p_i)$ for any station s located on ∂B_i, thus a new station placed at s^* produces the desired energy at p_i for $i = 1, 2$, that is, $\mathrm{E}(s^*, p_i) = \mathrm{E}(\{s_1, s_2\}, p_i)$. Hence assertion (1) is satisfied.

Consider a uniform power network $\mathcal{A}^*$ consisting of the stations s^*, s_1, and s_2 with no background noise. We have $\mathrm{SINR}_{\mathcal{A}^*}(s^*, p_i) = \frac{\mathrm{E}(s^*, p_i)}{\mathrm{E}(\{s_1, s_2\}, p_i)} = 1$ for $i = 1, 2$. Therefore, Lemma 3.3 (with s^*, $\mathcal{H}^*$ and 1 substituted for s_0, $\mathcal{H}_0$ and β respectively) guarantees that $\frac{\mathrm{E}(s^*, q)}{\mathrm{E}(\{s_1, s_2\}, q)} = \mathrm{SINR}_{\mathcal{A}^*}(s^*, q) \geq 1$ for all points $q \in \overline{p_1\, p_2}$, which means that $\mathrm{E}(s^*, q) \geq \mathrm{E}(\{s_1, s_2\}, q)$. Thus assertion (2) follows as well, completing the proof of Lemma 3.6. □

We now turn to describe the inductive step in the proof of Lemma 3.5.

Proof of Lemma 3.5. Assume by induction that the assertion of the theorem holds for $n \geq 2$ stations, i.e., that in a uniform power network with $n \geq 2$ stations and no background noise we have $\overline{p_1\, p_2} \subseteq \mathcal{H}_0$ for every $p_1, p_2 \in \mathcal{H}_0$. Now consider a uniform power network $\mathcal{A}$ with $n+1$ stations

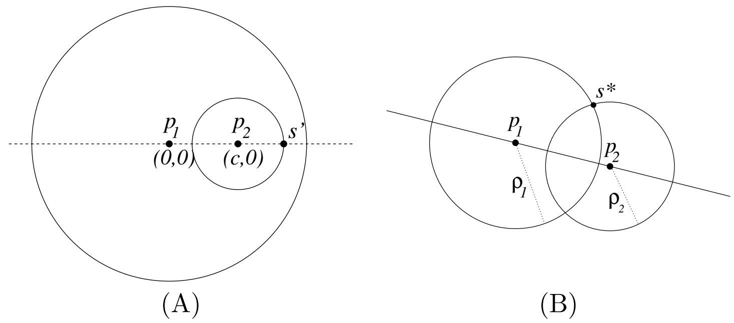


Figure 10: (A) The setting if B_2 is strictly contained in B_1. (B) The intersection point s^* is at distance ρ_i from p_i for $i = 1, 2$.

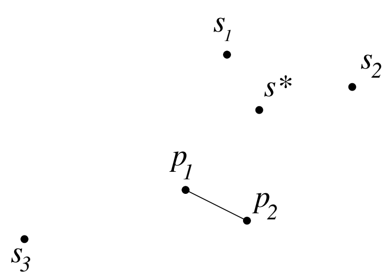


Figure 11: $\mathcal{A}^*$ is obtained from $\mathcal{A}$ by removing stations s_1 and s_2 and introducing station s^*.

$s_0, \ldots, s_n$ and no background noise. Let $p_1, p_2 \in \mathcal{H}_0$. Suppose that s_1 is closest to, say, p_1 among all stations $s_1, \ldots, s_n$. Since $p_1, p_2 \in \mathcal{H}_0$, we know that $\mathrm{E}(s_0, p_i) > \mathrm{E}(\{s_1, s_2\}, p_i)$ for $i = 1, 2$. Let s^* be the point whose existence is asserted by Lemma 3.6 .

Note that s^* must differ from s_0. This is because $\mathrm{E}(s^*, p_i) = \mathrm{E}(\{s_1, s_2\}, p_i)$ while $\mathrm{E}(s_0, p_i) > \mathrm{E}(\{s_1, s_2\}, p_i)$ for $i = 1, 2$, thus $\mathrm{dist}(s^*, p_i) > \mathrm{dist}(s_0, p_i)$.

Consider the n-station uniform power network $\mathcal{A}^*$ obtained from $\mathcal{A}$ by replacing s_1 and s_2 with a single station located at s^* (see Figure 11). Note that $\mathrm{I}_{\mathcal{A}^*}(s_0, p_i) = \mathrm{I}_{\mathcal{A}}(s_0, p_i)$ for $i = 1, 2$ and $\mathrm{I}_{\mathcal{A}^*}(s_0, q) \geq \mathrm{I}_{\mathcal{A}}(s_0, q)$ for all points $q \in \overline{p_1 \, p_2}$, hence $\mathrm{SINR}_{\mathcal{A}^*}(s_0, p_i) = \mathrm{SINR}_{\mathcal{A}}(s_0, p_i)$ for $i = 1, 2$ and $\mathrm{SINR}_{\mathcal{A}^*}(s_0, q) \leq \mathrm{SINR}_{\mathcal{A}}(s_0, q)$. By the inductive hypothesis, $\mathrm{SINR}_{\mathcal{A}^*}(s_0, q) \geq \beta$ for all points $q \in \overline{p_1 \, p_2}$, therefore $\mathrm{SINR}_{\mathcal{A}}(s_0, q) \geq \beta$ and s_0 is heard at q in $\mathcal{A}$. It follows that every $q \in \overline{p_1 \, p_2}$ belongs to $\mathcal{H}_0$ in $\mathcal{A}$, which establishes the assertion and completes the proof of Lemma 3.5. □

3.4 Convexity with background noise

Our goal in this section is to complete the proof of Theorem 1 in the Euclidean plane, by extending the convexity proof to the case with background noise. Formally, we show the following.

Lemma 3.7. *The reception zones in an n-station uniform power network $\mathcal{A} = \langle d = 2, S, \bar{1}, N, \beta, \alpha \rangle$, where $N > 0$, are convex.*

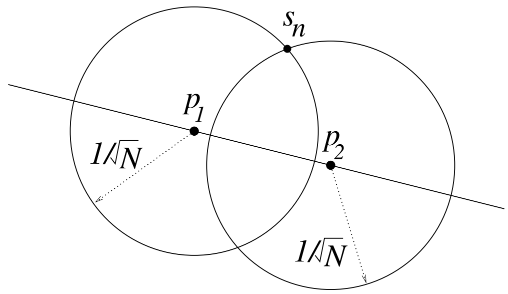


Figure 12: $\mathcal{A}'$ is obtained from $\mathcal{A}$ by omitting the background noise N and introducing station s_n.

Proof. Let p_1 and p_2 be points in $\mathbb{R}^2$ and suppose that s_0 is heard at p_1 and p_2 in $\mathcal{A}$. Let B_1 and B_2 be the balls of radius $1/\sqrt{N}$ centered at p_1 and p_2, respectively. Note that $\mathrm{SINR}_{\mathcal{A}}(s_0, p_i) \geq \beta \geq 1$ implies that $\mathrm{E}(s_0, p_i) > N$, thus $\mathrm{dist}(s_0, p_i) < 1/\sqrt[\alpha]{N}$ for $i = 1, 2$. Therefore $\mathrm{dist}(p_1, p_2) < 2/\sqrt[\alpha]{N}$ and ∂B_1 and ∂B_2 must intersect.

We construct an $(n+1)$-station uniform power network $\mathcal{A}'$ from $\mathcal{A}$ by locating a new station s_n (with transmitting power $\psi_n = 1$ like all other stations) in an intersection point of ∂B_1 and ∂B_2 and omitting the background noise (see Figure 12). Clearly, $\mathrm{E}(s_n, p_i) = N$ for $i = 1, 2$. In particular, this means that $s_n \neq s_0$ as $\mathrm{E}(s_0, p_i) > N$. Since $\mathrm{dist}(s_n, p_1) = \mathrm{dist}(s_n, p_2) = 1/\sqrt{N}$, it follows that $\mathrm{dist}(s_n, q) \leq 1/\sqrt[\alpha]{N}$ for all points $q \in \overline{p_1\, p_2}$, hence $\mathrm{E}(s_n, q) \geq N$. Therefore $\mathrm{SINR}_{\mathcal{A}'}(s_0, p_i) = \mathrm{SINR}_{\mathcal{A}}(s_0, p_i)$ for $i = 1, 2$ and $\mathrm{SINR}_{\mathcal{A}}(s_0, q) \geq \mathrm{SINR}_{\mathcal{A}'}(s_0, q)$ for all points $q \in \overline{p_1\, p_2}$. Since $\mathcal{A}'$ has no background noise, we may employ Lemma 3.5 to conclude that $\mathrm{SINR}_{\mathcal{A}'}(s_0, q) \geq \beta$ for all points $q \in \overline{p_1\, p_2}$. The assertion follows. □

This completes the proof of Theorem 1 for the case where $d = 2$.

3.5 From the Euclidean plane to a d-dimensional space

Our goal in this section is to extend the convexity result to higher dimensional spaces, and show the following.

Lemma 3.8. *The reception zones in a uniform power network $\mathcal{A} = \langle d, S, \bar{1}, N, \beta, \alpha \rangle$, where $d \geq 2$ and $N \geq 0$, are convex in the d-dimensional Euclidean space.*

Proof. Let p_1 and p_2 be some points in $\mathbb{R}^d$ and suppose that s_0 is heard at p_1 and p_2 in $\mathcal{A}$, i.e., $p_1, p_2 \in \mathcal{H}_0$. By Lemma 2.3, we may assume without loss of generality that $p_1 = (0, 0, ..., 0)$ and $p_2 = (c, 0, ..., 0)$, where $c \neq 0$. Note that the straight line that goes through p_1 and p_2 is the horizontal axis, i.e., the line $L^* = \{(x, 0, ..., 0) \mid x \in \mathbb{R}\}$.

We construct a new uniform power network $\mathcal{A}' = \langle 2, S', \bar{1}, N, \beta, \alpha \rangle$ in which all stations are deployed on the Euclidean plane. The network $\mathcal{A}'$ is obtained from $\mathcal{A}$ by rotating each station s_i around L^* until it reaches the plane $\{(x, y, 0, ..., 0) \mid x, y \in \mathbb{R}\}$. Note that if some station s_i is not located on this plane, i.e., there exists $3 \leq j \leq d$ such that $x^i_j \neq 0$, then it can be relocated to a new location s'_i on the plane by rotating it around L^* so that $\text{dist}(s_i, q) = \text{dist}(s'_i, q)$, for every $q \in L^*$. More formally, the location of s'_i is $(a_i, b_i) \in \mathbb{R}^2$, where $a_i = x^i_1$ and $b_i = \sqrt{\sum_{j=2}^{d}(x^i_j)^2}$, for every $i = 0, 1, ..., n-1$. (Recall that $s_i = (x^i_1, x^i_2, ...x^i_d)$.) It follows that

$$\text{dist}(s_i, q) = \sqrt{\left(x^i_1 - x\right)^2 + \sum_{j=2}^{d}\left(x^i_j\right)^2} = \sqrt{(a_i - x)^2 + b_i^2} = \text{dist}(s'_i, q)$$

for every $i = 0, 1, \ldots, n-1$ and every $q = (x, 0, \ldots, 0) \in L^*$, and in particular, for every $q \in \overline{p_1\, p_2}$. This implies that

$$\text{SINR}_{\mathcal{A}}(s_0, q) = \text{SINR}_{\mathcal{A}'}(s'_0, q) \tag{5}$$

for all points $q \in \overline{p_1\, p_2}$.

The stations in the uniform power network $\mathcal{A}'$ are deployed in the Euclidean plane ($d = 2$), hence we already know that the reception zone $\mathcal{H}'_0$ of s'_0 is convex. This implies that $\text{SINR}_{\mathcal{A}'}(s'_0, q) \geq \beta$, for all points $q \in \overline{p_1\, p_2}$. By applying Eq. (5), we conclude that $\text{SINR}_{\mathcal{A}}(s_0, q) \geq \beta$ for all points $q \in \overline{p_1\, p_2}$, hence $\overline{p_1\, p_2} \subseteq \mathcal{H}_0$ and the assertion follows. □

This completes the proof of Theorem 1 for the d-dimensional space for any $d \geq 2$. The one-dimensional case ($d = 1$) is trivial to analyze, and in particular, follows from Lemma 3.1.

4 The fatness of the reception zones

In Section 3 we showed that the reception zone of each station in a uniform power network is convex. In this section we develop a deeper understanding of the shape of the reception zones by analyzing their fatness. Consider a uniform power network $\mathcal{A} = \langle d, S, \bar{1}, N, \beta, \alpha \rangle$, where $S = \{s_0, \ldots, s_{n-1}\}$ and $\alpha > 0$ and $\beta > 1$ are constants[5]. We focus on s_0 and assume that its location is not shared by any other station (otherwise, its reception zone is $\mathcal{H}_0 = \{s_0\}$). In Section 4.1 we establish explicit bounds on the radii $\Delta(s_0, \mathcal{H}_0)$ and $\delta(s_0, \mathcal{H}_0)$. These bounds imply that $\varphi(s_0, \mathcal{H}_0) = O(\sqrt[\alpha]{n})$. This is improved in Section 4.2, where we show that $\varphi(s_0, \mathcal{H}_0) = O(1)$, thus establishing Theorem 2.

4.1 Explicit bounds

Our goal in this section is to establish an explicit lower bound on $\delta(s_0, \mathcal{H}_0)$ and an explicit upper bound on $\Delta(s_0, \mathcal{H}_0)$. Since $\mathcal{H}_0$ is compact and convex, it follows that there exist some points on its boundary, $q_\delta, q_\Delta \in \partial\mathcal{H}_0$, such that $\text{dist}(s_0, q_\delta) = \delta(s_0, \mathcal{H}_0)$ and $\text{dist}(s_0, q_\Delta) = \Delta(s_0, \mathcal{H}_0)$. In fact,

[5] Unlike the convexity proof presented in Section 3, which holds for any $\beta \geq 1$, the analysis presented in the current section is only suitable for β being a constant strictly greater than 1. The special case when $\beta = 1$ is a bit more complex and its discussion is not presented here.

we may redefine $\delta(s_0, \mathcal{H}_0)$ as the distance from s_0 to a closest point on $\partial\mathcal{H}_0$ and $\Delta(s_0, \mathcal{H}_0)$ as the distance from s_0 to a farthest point on $\partial\mathcal{H}_0$. To avoid cumbersome notation, we assume a two-dimensional space ($d = 2$) throughout this section; the proof is trivially generalized to arbitrary choices of d.

Fix $\kappa = \min\{\text{dist}(s_0, s_i) \mid i > 0\}$. An extreme scenario for establishing a lower bound on $\delta(s_0, \mathcal{H}_0)$ (namely, a scenario yielding the smallest δ value) would be to place s_0 at $(0, 0)$ and all other $n-1$ stations at $(\kappa, 0)$. This introduces the uniform power network $\mathcal{A}_\delta = \langle 2, \{(0,0), (\kappa, 0), \ldots, (\kappa, 0)\}, \bar{1}, N, \beta, \alpha\rangle$. The point q_δ whose distance to s_0 realizes $\delta(s_0, \mathcal{H}_0)$ is thus located at $(d, 0)$ for some $0 < d < \kappa$. On the other hand, an extreme scenario for establishing an upper bound on $\Delta(s_0, \mathcal{H}_0)$ (namely, a scenario yielding the largest Δ value) would be to place s_0 in $(0, 0)$, s_1 in $(\kappa, 0)$, and all other $n-2$ stations in $(\infty, 0)$, so that their energy at the vicinity of s_0 is ignored. This introduces the uniform power network $\mathcal{A}_\Delta = \langle 2, \{(0,0), (\kappa, 0), (\infty, 0), \ldots, (\infty, 0)\}, \bar{1}, N, \beta, \alpha\rangle$. The point q_Δ whose distance to s_0 realizes $\Delta(s_0, \mathcal{H}_0)$ is thus located at $(-D, 0)$ for some $D > 0$.

For the sake of analysis, we replace the background noise N in the above scenarios with a new station s_n located at $(\kappa, 0)$ whose power is $\psi_n = N \cdot \kappa^\alpha$. More formally, the uniform power network $\mathcal{A}_\delta$ is replaced by the network

$$\mathcal{A}'_\delta = \langle 2, \{(0,0), (\kappa, 0), \ldots, (\kappa, 0), (\kappa, 0)\}, (1, \ldots, 1, N \cdot \kappa^\alpha), 0, \beta, \alpha\rangle$$

and the uniform power network $\mathcal{A}_\Delta$ is replaced by the network

$$\mathcal{A}'_\Delta = \langle 2, \{(0,0), (\kappa, 0), (\infty, 0) \ldots, (\infty, 0), (\kappa, 0)\}, (1, \ldots, 1, N \cdot \kappa^\alpha), 0, \beta, \alpha\rangle \ .$$

Note that the energy of the new station s_n at point $(x, 0)$ satisfies $\mathrm{E}(s_n, (x, 0)) > N$ for all $0 < x < \kappa$; $\mathrm{E}(s_n, (x, 0)) = N$ for $x = 0$; and $\mathrm{E}(s_n, (x, 0)) < N$ for all $x < 0$.

Therefore, the value of $\delta(s_0, \mathcal{H}_0)$ under $\mathcal{A}'_\delta$ is smaller than that under $\mathcal{A}_\delta$, and the value of $\Delta(s_0, \mathcal{H}_0)$ under $\mathcal{A}'_\Delta$ is greater than that under $\mathcal{A}_\Delta$.

To establish a lower bound on $\delta(s_0, \mathcal{H}_0)$ in the context of $\mathcal{A}'_\delta$, we would like to compute the value of $\check{x} > 0$ that solves the equation

$$\mathrm{SINR}_{\mathcal{A}'_\delta}(s_0, (\check{x}, 0)) \ = \ \frac{\check{x}^{-\alpha}}{(n - 1 + N \cdot \kappa^\alpha)(\kappa - \check{x})^{-\alpha}} \ = \ \beta \ .$$

Rearranging, we get $(\kappa - \check{x})^\alpha = \check{x}^\alpha \beta(n - 1 + N \cdot \kappa^\alpha)$, yielding $\check{x} = \frac{\kappa}{\sqrt[\alpha]{\beta(n-1+N\cdot\kappa^\alpha)}+1}$, hence $\delta(s_0, \mathcal{H}_0) \geq \frac{\kappa}{\sqrt[\alpha]{\beta(n-1+N\cdot\kappa^\alpha)}+1}$.

To establish an upper bound on $\Delta(s_0, \mathcal{H}_0)$ in the context of $\mathcal{A}'_\Delta$, we would like to compute the value of $\hat{x} > 0$ that solves the equation

$$\mathrm{SINR}_{\mathcal{A}'_\Delta}(s_0, (-\hat{x}, 0)) \ = \ \frac{\hat{x}^{-\alpha}}{(1 + N \cdot \kappa^\alpha)(\kappa + \hat{x})^{-\alpha}} \ = \ \beta \ .$$

Rearranging, $(\kappa+\hat{x})^\alpha = \hat{x}^\alpha \beta(1+N\cdot\kappa^\alpha)$, or $\hat{x} = \frac{\kappa}{\sqrt[\alpha]{\beta(1+N\cdot\kappa^\alpha)}-1}$, hence $\Delta(s_0, \mathcal{H}_0) \leq \frac{\kappa}{\sqrt[\alpha]{\beta(1+N\cdot\kappa^\alpha)}-1}$.

The fatness parameter of $\mathcal{H}_0$ with respect to s_0 thus satisfies

$$\varphi(s_0, \mathcal{H}_0) \leq \frac{\kappa}{\sqrt[\alpha]{\beta(1+N\cdot\kappa^\alpha)}-1} \Big/ \frac{\kappa}{\sqrt[\alpha]{\beta(n-1+N\cdot\kappa^\alpha)}+1} \leq \frac{\sqrt[\alpha]{\beta(n-1)}+1}{\sqrt[\alpha]{\beta}-1} = O\left(\sqrt[\alpha]{n}\right) ,$$

where the second inequality makes use of Obs. 2.5. This yields the following.

Theorem 4.1. *In a uniform energy network* $\mathcal{A} = \langle 2, S, \bar{1}, N, \beta, \alpha \rangle$, *where* $S = \{s_0, \ldots, s_{n-1}\}$ *and* $\alpha > 0$ *and* $\beta > 1$ *are constants, if* $\kappa = \min\{\mathrm{dist}(s_0, s_i) \mid i > 0\} > 0$, *then*

$$\delta(s_0, \mathcal{H}_0) \geq \frac{\kappa}{\sqrt[\alpha]{\beta(n-1+N\cdot\kappa^\alpha)}+1} , \Delta(s_0, \mathcal{H}_0) \leq \frac{\kappa}{\sqrt[\alpha]{\beta(1+N\cdot\kappa^\alpha)}-1} ,$$

and $\varphi(s_0, \mathcal{H}_0) = O(\sqrt[\alpha]{n})$.

4.2 An improved bound on the fatness parameter

In this section we prove Theorem 2 by establishing the following theorem.

Theorem 4.2. *The fatness parameter of* $\mathcal{H}_0$ *with respect to* s_0 *satisfies*

$$\varphi(s_0, \mathcal{H}_0) \ \leq \ \frac{\sqrt[\alpha]{\beta}+1}{\sqrt[\alpha]{\beta}-1}$$

which is $O(1)$ *for every constant path-loss parameter* $\alpha > 0$ *and reception threshold* $\beta > 1$.

Theorem 4.2 is proved in three steps. First, in Section 4.2.1 we bound the ratio Δ/δ in a setting of two stations in a one-dimensional space. This is used in Section 4.2.2 to establish the desired bound for a special type of uniform power networks called *positive collinear* networks. We conclude in Section 4.2.3, where we reduce the general case to the case of positive collinear networks.

4.2.1 Two stations in a one-dimensional space

Let $\mathcal{A} = \langle 1, \{s_0, s_1\}, (1, \psi_1), N, \beta, \alpha \rangle$ be a network consisting of two stations s_0, s_1 embedded in the one-dimensional space $\mathbb{R}$ with no background noise (i.e., $N = 0$). Assume without loss of generality that s_0 is located at $a_0 = 0$ and s_1 is located at $a_1 = 1$ (due to Lemma 2.3). Suppose that s_0 uses transmitting power $\psi_0 = 1$ while the transmitting power of s_1 is any $\psi_1 \geq 1$. Let $\mu_r = \max\{p > 0 \mid \mathrm{SINR}_{\mathcal{A}}(s_0, p) \geq \beta\}$ and let $\mu_l = \min\{p < 0 \mid \mathrm{SINR}_{\mathcal{A}}(s_0, p) \geq \beta\}$ (see Figure 13). It is easy to verify that $\mathcal{H}_0 = [\mu_l, \mu_r]$ and that $\delta = \delta(s_0, \mathcal{H}_0) = \mu_r$ and $\Delta = \Delta(s_0, \mathcal{H}_0) = -\mu_l$.

Lemma 4.3. *The network* $\mathcal{A}$ *satisfies* $\varphi(s_0, \mathcal{H}_0) = \Delta/\delta \leq \frac{\sqrt[\alpha]{\beta}+1}{\sqrt[\alpha]{\beta}-1}$, *with equality attained when* $\psi_1 = 1$.

Proof. The boundary points μ_r and μ_l of $\mathcal{H}_0$ are the solutions to the equation

$$\frac{1/|x|^\alpha}{\psi_1/(1-x)^\alpha} = \beta, \qquad x < 1,$$

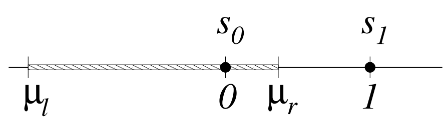


Figure 13: The embedding of s_0 and s_1 in a one-dimensional space.

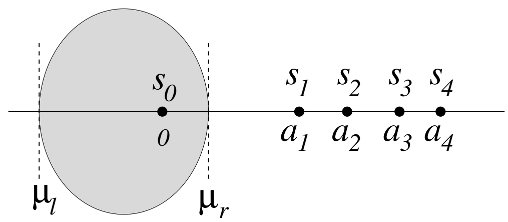


Figure 14: A positive collinear network.

leading to $\frac{1-x}{|x|} = (\beta\psi_1)^{1/\alpha}$, or, $x + (\beta\psi_1)^{1/\alpha}|x| = 1$. Solving this linear equation separately for positive and negative x yields, respectively, the solutions

$$\mu_r = \frac{1}{1+(\beta\psi_1)^{1/\alpha}} \quad \text{and} \quad \mu_l = \frac{1}{1-(\beta\psi_1)^{1/\alpha}} .$$

Therefore the ratio Δ/δ satisfies

$$\frac{\Delta}{\delta} = \frac{-\mu_l}{\mu_r} = \frac{\sqrt[\alpha]{\beta\psi_1}+1}{\sqrt[\alpha]{\beta\psi_1}-1} \leq \frac{\sqrt[\alpha]{\beta}+1}{\sqrt[\alpha]{\beta}-1} ,$$

where the last inequality holds since $\alpha > 0$, $\beta > 1$ and $\psi_1 \geq 1$. The Lemma follows. □

4.2.2 Positive collinear networks

In this section we switch back to the Euclidean plane $\mathbb{R}^2$ and consider a special type of uniform power networks. A network $\mathcal{A} = \langle 2, \{s_0, \ldots, s_{n-1}\}, \bar{1}, N, \beta, \alpha \rangle$ is said to be *positive collinear* if $s_0 = (0,0)$ and $s_i = (a_i, 0)$ for some $a_i > 0$ for every $1 \leq i \leq n-1$. Positive collinear networks play an important role in the subsequent analysis due to the following lemma. (Refer to Figure 14 for illustration.)

Lemma 4.4. *Let $\mathcal{A}$ be a positive collinear uniform power network. Fix $\mu_r = \max\{r > 0 \mid \mathrm{SINR}_{\mathcal{A}}(s_0,(r,0)) \geq \beta\}$ and $\mu_l = \min\{r < 0 \mid \mathrm{SINR}_{\mathcal{A}}(s_0,(r,0)) \geq \beta\}$. Then*
(1) $\delta(s_0, \mathcal{H}_0) = \mu_r$,
(2) $\Delta(s_0, \mathcal{H}_0) = -\mu_l$, and
(3) $\varphi(s_0, \mathcal{H}_0) \leq \frac{\sqrt[\alpha]{\beta}+1}{\sqrt[\alpha]{\beta}-1}$.

Proof. First, we argue that the reception zone $\mathcal{H}_0$ of s_0 in the positive collinear network $\mathcal{A}$ is contained in the infinite vertical strip defined by $\mu_l \leq x \leq \mu_r$. To see why this is true, suppose, towards contradiction, that the point $(x,y) \in \mathcal{H}_0$ for some $x > \mu_r$ or $x < \mu_l$. By symmetry considerations, the point $(x, -y)$ is also in $\mathcal{H}_0$. By the convexity of $\mathcal{H}_0$, it follows that $(x, 0) \in \mathcal{H}_0$, in contradiction to the definitions of μ_r and μ_l. We thus have the following.

Claim 4.5. *If $(x, y) \in \mathcal{H}_0$, then $\mu_l \leq x \leq \mu_r$.*

To prove assertion (1) of the lemma, we show that the ball of radius μ_r centered at s_0 is contained in $\mathcal{H}_0$. In fact, by the convexity of $\mathcal{H}_0$, it is sufficient to show that the point $p(\theta) = (\mu_r \cos\theta, \mu_r \sin\theta)$ is in $\mathcal{H}_0$ for all $0 \leq \theta \leq \pi$. Since the network is positive collinear, it follows that $\mathrm{I}_{\mathcal{A}}(s_0, p(\theta))$ attains its maximum for $\theta = 0$. Therefore the fact that $p(0) = (\mu_r, 0) \in \mathcal{H}_0$ implies that $p(\theta) \in \mathcal{H}_0$ for all $0 \leq \theta \leq \pi$ as desired. Assertion (1) follows.

Next, we show that Δ is realized by the point $(\mu_l, 0)$. Indeed, by the triangle inequality, all points at distance k from s_0 are at distance at most $k + a_i$ from $s_i = (a_i, 0)$, with equality attained for the point $(-k, 0)$. Thus the minimum interference to s_0 under $\mathcal{A}$ among all points at distance k from s_0 is attained at the point $(-k, 0)$. Therefore, by the definition of μ_l, there cannot exist any point $p \in \mathcal{H}_0$ such that $\mathrm{dist}(p, s_0) > -\mu_l$. Assertion (2) follows.

It remains to establish assertion (3). Fix $c = \min\{a_i \mid 1 \leq i \leq n-1\}$, that is, the leftmost station other than s_0 is located at $(c, 0)$. Clearly, $\mu_r < c$. Denote the energy of station s_i at $(\mu_r, 0)$ by $\mathcal{E}_i = \mathrm{E}(s_i, (\mu_r, 0)) = (a_i - \mu_r)^{-\alpha}$. We construct a new $(n+1)$-station network $\mathcal{A}' = \langle 2, S', \psi', 0, \beta, \alpha\rangle$ consisting of s_0 and n new stations $s'_1, \ldots, s'_n$, all located at $(c, 0)$. We set the transmitting power ψ'_i of the new stations s'_i to

$$\psi'_i = \begin{cases} \mathcal{E}_i \cdot (c - \mu_r)^{\alpha} & \text{for } 1 \leq i \leq n-1; \text{ and} \\ N \cdot (c - \mu_r)^{\alpha} & \text{for } i = n. \end{cases}$$

This ensures that the energy produced by these stations at $(\mu_r, 0)$ is

$$\mathrm{E}(s'_i, (\mu_r, 0)) = \begin{cases} \mathcal{E}_i & \text{for } 1 \leq i \leq n-1, \text{ and} \\ N & \text{for } i = n. \end{cases}$$

The network $\mathcal{A}'$ falls into the setting of Section 4.2.1: the stations $s'_1, \ldots, s'_n$ share the same location, thus they can be considered as a single station $\hat{s}_1$ with transmitting power $\hat{\psi}_1 = \sum_{i=1}^{n} \psi'_i$. Define $\mu'_r = \max\{r > 0 \mid \mathrm{SINR}_{\mathcal{A}'}(s_0, (r, 0)) \geq \beta\}$ and $\mu'_l = \min\{r < 0 \mid \mathrm{SINR}_{\mathcal{A}'}(s_0, (r, 0)) \geq \beta\}$, so that the restriction of the reception zone of s_0 under $\mathcal{A}'$ to the x-axis is $[\mu'_l, \mu'_r]$. Lemma 4.3 implies that $-\mu'_l/\mu'_r \leq \frac{\sqrt[\alpha]{\beta}+1}{\sqrt[\alpha]{\beta}-1}$. The remainder of the proof relies on establishing the following two bounds, relating the networks $\mathcal{A}$ and $\mathcal{A}'$.

(A1) $\mathrm{SINR}_{\mathcal{A}'}(s_0, (r, 0)) \leq \mathrm{SINR}_{\mathcal{A}}(s_0, (r, 0))$ for all $\mu_r \leq r < c$; and

(A2) $\mathrm{SINR}_{\mathcal{A}'}(s_0, (r, 0)) \geq \mathrm{SINR}_{\mathcal{A}}(s_0, (r, 0))$ for all $r \leq \mu_r$, $r \neq 0$.

By bound (A1), $\mathrm{SINR}_{\mathcal{A}'}(s_0, (r, 0)) \leq \mathrm{SINR}_{\mathcal{A}}(s_0, (r, 0)) < \beta$, for every $r \in (\mu_r, c)$, thus $\mu'_r \leq \mu_r$. (In fact, $\mu'_r = \mu_r$, since the construction of $\mathcal{A}'$ implies that $\mu'_r \geq \mu_r$ as $\mathrm{SINR}_{\mathcal{A}'}(s_0, (\mu_r, 0)) =$

$\mathrm{SINR}_{\mathcal{A}}(s_0,(\mu_r,0)) \geq \beta$.) By bound (A2), it follows, in particular, that $\mathrm{SINR}_{\mathcal{A}'}(s_0,(\mu_l,0)) \geq \mathrm{SINR}_{\mathcal{A}}(s_0,(\mu_l,0)) \geq \beta$, thus, $\mu_l' \leq \mu_l$. Therefore, $-\mu_l/\mu_r \leq -\mu_l'/\mu_r' \leq \frac{\sqrt[\alpha]{\beta}+1}{\sqrt[\alpha]{\beta}-1}$, which completes the proof of Lemma 4.4.

To establish bounds (A1) and (A2), consider some point $p = (r,0)$, where $r < c$, $r \neq 0$. For every $1 \leq i \leq n-1$, we have

$$\mathrm{E}(s_i,p) = \frac{1}{(a_i - r)^\alpha} \ , \quad \text{while} \quad \mathrm{E}(s_i',p) = \frac{\psi_i'}{(c-r)^\alpha} = \frac{(c-\mu_r)^\alpha}{(c-r)^\alpha (a_i - \mu_r)^\alpha} \ .$$

Comparing these two expressions, we get $\mathrm{E}(s_i,p) \geq \mathrm{E}(s_i',p)$, or equivalently, $(c-r)(a_i-\mu_r) \geq (c-\mu_r)(a_i - r)$. Rearranging, we get $ca_i - c\mu_r - a_i r + r\mu_r \geq ca_i - cr - a_i\mu_r + r\mu_r$, or

$$\mu_r(a_i - c) \ \geq \ r(a_i - c) \ ,$$

where the last inequality holds if and only $a_i = c$, which, by definition, implies that $\mathrm{E}(s_i,p) = \mathrm{E}(s_i',p)$, or $\mu_r \geq r$. Therefore the contribution of s_i' to the interference to s_0 at $p = (0,r)$ is not larger than that of s_i as long as $r \leq \mu_r$ and not smaller than that of s_i as long as $\mu_r \leq r < c$. On the other hand, the energy of s_n' at $p = (r,0)$ satisfies $\mathrm{E}(s_n',p) \leq N$ for all $c \leq \mu_r$ and $\mathrm{E}(s_n',p) \geq N$ for all $\mu_r \leq r < c$. Bounds (A1) and (A2) follow. □

4.2.3 General uniform power networks in d-dimensional space

We are now ready to prove the main theorem of Section 4.

Proof of Theorem 4.2. Consider an arbitrary uniform power network $\mathcal{A} = \langle d, S, \bar{1}, N, \beta, \alpha \rangle$, where $S = \{s_0, \ldots, s_{n-1}\}$ and $\beta > 1$ is a constant. We employ Lemma 2.3 to assume without loss of generality that s_0 is located at $(0,...,0)$ and that $\max\{\mathrm{dist}(s_0,q) \mid q \in \mathcal{H}_0\}$ is realized by a point $q = (-\Delta, 0, ..., 0)$ on the negative x-axis. We now construct a new positive collinear uniform power network $\mathcal{A}' = \langle d, \{s_0, s_1', \ldots, s_{n-1}'\}, \bar{1}, N, \beta, \alpha \rangle$, obtained from $\mathcal{A}$ by rotating each station s_i around the point q until it reaches the Euclidean plane at the positive x-axis (see Figure 15). More formally, the location of s_0 remains unchanged and $s_i' = (a_i', 0)$, where $a_i' = \mathrm{dist}(s_i,q) - \Delta$ for every $1 \leq i \leq n-1$. Since s_0 is heard at q under $\mathcal{A}$, it follows that $\Delta = \mathrm{dist}(s_0,q) < \mathrm{dist}(s_i,q)$ for every $1 \leq i \leq n-1$, hence $a_i' > 0$ and $\mathcal{A}'$ is a positive collinear network. Clearly, $\mathrm{dist}(s_i',q) = \mathrm{dist}(s_i,q)$ for every $1 \leq i \leq n-1$.

Let $\mathcal{H}_0'$ denote the reception zone of s_0 under $\mathcal{A}'$. Fix $\delta' = \max\{r > 0 \mid B(s_0,r) \subseteq \mathcal{H}_0'\}$ and $\Delta' = \min\{r > 0 \mid B(s_0,r) \supseteq \mathcal{H}_0'\}$. Let $\mu_r' = \max\{r > 0 \mid \mathrm{SINR}_{\mathcal{A}'}(s_0,(r,0,...,0)) \geq \beta\}$ and let $\mu_l' = \min\{r < 0 \mid \mathrm{SINR}_{\mathcal{A}'}(s_0,(r,0,...,0)) \geq \beta\}$. Lemma 4.4 guarantees that $\delta' = \mu_r'$, $\Delta' = -\mu_l'$, and $\frac{\Delta'}{\delta'} \leq \frac{\sqrt[\alpha]{\beta}+1}{\sqrt[\alpha]{\beta}-1}$. We establish the proof of Theorem 4.2 by showing that $\Delta' = \Delta$ and $\delta' \leq \delta$. The former is a direct consequence of Lemma 4.4; since $\mathrm{SINR}_{\mathcal{A}'}(s_0,q) = \mathrm{SINR}_{\mathcal{A}}(s_0,q) = \beta$, it follows that $\max\{\mathrm{dist}(s_0,p) \mid p \in \mathcal{H}_0'\}$ is realized at $p = q$.

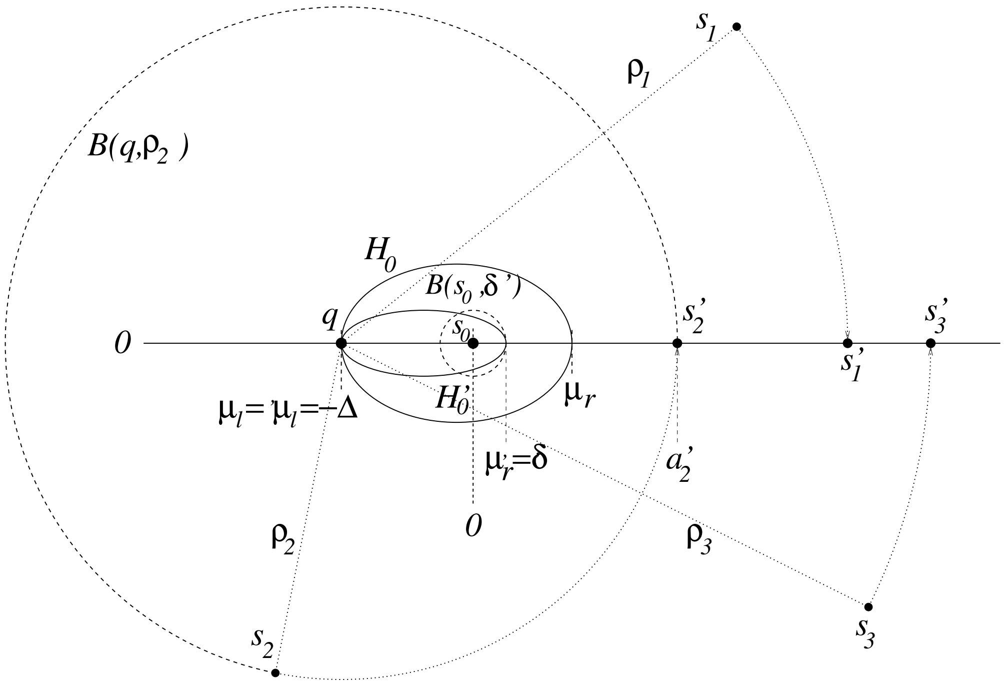


Figure 15: $\mathcal{A}'$ is obtained from $\mathcal{A}$ by relocating each station s_i on the x-axis.

It remains to prove that $\delta' \leq \delta$. We do so by showing that $B(s_0, \delta') \subseteq \mathcal{H}_0$. Fix $\rho_i = \text{dist}(s_i, q)$ for every $1 \leq i \leq n-1$. We argue that the ball $B(s_0, \delta')$ is strictly contained in the ball $B(q, \rho_i)$ for every $1 \leq i \leq n-1$. To see why this is true, observe that $-\Delta < 0 < \delta' = \mu_r' < a_i'$, hence the ball centered at $q = (-\Delta, 0, ..., 0)$ of radius $\rho_i = \Delta + a_i'$ strictly contains the ball of radius δ' centered at $s_0 = (0, ..., 0)$.

Consider an arbitrary point $p \in B(s_0, \delta')$. We can now rewrite

$$\text{dist}(s_i', (\delta', 0, ..., 0)) = a_i' - \delta' = \min\{\text{dist}(t, t') \mid t \in B(s_0, \delta'), t' \in \partial B(q, \rho_i)\}$$

for every $1 \leq i \leq n-1$. Recall that $s_i \in \partial B(q, \rho_i)$, thus $\text{dist}(s_i, p) \geq \text{dist}(s_i', (\delta', 0, ..., 0))$. Therefore $\text{I}_{\mathcal{A}}(s_0, p) \leq \text{I}_{\mathcal{A}'}(s_0, (\delta', 0, ..., 0))$ and $\text{SINR}_{\mathcal{A}}(s_0, p) \geq \text{SINR}_{\mathcal{A}'}(s_0, (\delta', 0)) = \beta$. It follows that $p \in \mathcal{H}_0$, which completes the proof. □

5 Handling approximate point location queries

Our goal in this section is to prove Theorem 3, namely, to show how a data structure supporting approximate point location queries is constructed. The input to the algorithm constructing our data structure is a uniform power network $\mathcal{A} = \langle 2, S, \bar{1}, N, \beta, \alpha \rangle$ given by specifying the coordinates of each station s_i, the background noise N, the reception threshold β, and the path-loss parameter

α. Since point location queries can trivially be answered in constant time for a constant number of stations, we assume hereafter that $n = |S|$ is large, and in particular, $n \geq 3$.

In fact, our technique for handling approximate point location queries is suitable for a more general class of zones (and diagrams). Let $\mathcal{Z}$ be a convex thick set in the euclidean plane and suppose that we are given an internal point s of $\mathcal{Z}$, a lower bound $\tilde{\delta}$ on $\delta(s, \mathcal{Z})$, and an upper bound $\tilde{\Delta}$ on $\Delta(s, \mathcal{Z})$. Moreover, suppose that we are given access to an oracle $\mathcal{O} : \mathbb{R}^2 \to \{0, 1\}$ for membership in $\mathcal{Z}$. Let $0 < \epsilon < 1$ be some predetermined performance parameter and fix $\rho = \tilde{\Delta}/\tilde{\delta}$.

Lemma 5.1. *By making $O(\rho^3\epsilon^{-1})$ calls to $\mathcal{O}$ in a preprocessing stage, we construct a data structure* `QDS` *of size $O(\rho^3\epsilon^{-1})$.* `QDS` *imposes a 3-wise partition $\bar{\mathcal{Z}} = \langle \mathcal{Z}^+, \mathcal{Z}^-, \mathcal{Z}^? \rangle$ of the Euclidean plane $\mathbb{R}^2$ (that is, the zones in $\bar{\mathcal{Z}}$ are pair-wise disjoint and $\mathbb{R}^2 = \mathcal{Z}^+ \cup \mathcal{Z}^- \cup \mathcal{Z}^?$) such that:*
(1) $\mathcal{Z}^+ \subseteq \mathcal{Z}$;
(2) $\mathcal{Z}^- \cap \mathcal{Z} = \emptyset$; and
(3) $\mathcal{Z}^?$ is bounded and its area is at most an ϵ-fraction of the area of $\mathcal{Z}$.
Given a query point $p \in \mathbb{R}^2$, it is possible to extract from `QDS`*, in constant time, the zone in $\bar{\mathcal{Z}}$ to which p belongs.*

Notice that we make the standard assumption that the size of the memory is measured in terms of the number of memory *words* being used, where a memory word can store a single real number (e.g., a point coordinate). It is also assumed that basic arithmetic operations on words and accessing a specified entry in a vector are performed in constant time.

In Section 5.1 we describe the construction of `QDS` and establish Lemma 5.1. In Section 5.2 we explain how the reception zones and the SINR diagram fall into the above framework and establish Theorem 3.

5.1 The construction of `QDS`

In this section we describe the construction of the data structure `QDS`. Let γ be a positive real to be determined later on. `QDS` is based upon imposing a γ-spaced *grid*, denoted by G_γ, on the Euclidean plane. The notions of grid *columns*, *rows*, *vertices*, *edges*, and *cells* are defined in the natural manner. We assume that G_γ is aligned so that the point s is a grid *vertex*.

A vertex of G_γ is said to be *internal* if it belongs to $\mathcal{Z}$; otherwise, it is said to be *external*. A grid cell is called *internal* (respectively, *external*) if all its (four) vertices are internal (resp., external). If the cell has at least one internal vertex and at least one external vertex, then it is said to be *mixed*. The convexity of $\mathcal{Z}$ ensures that an internal cell is fully contained in $\mathcal{Z}$. By definition, a mixed cell has a non-empty intersection with both $\mathcal{Z}$ and $\mathbb{R}^2 - \mathcal{Z}$. An external cell always intersects $\mathbb{R}^2 - \mathcal{Z}$, but it may also intersect $\mathcal{Z}$ which means that $\partial\mathcal{Z}$ has two intersection points with (at least) one of its edges. A *mixed* edge is an edge with one internal vertex and one external vertex. Therefore a mixed cell can be redefined as a cell admitting some (at least two) mixed edges. Refer to Figure 16

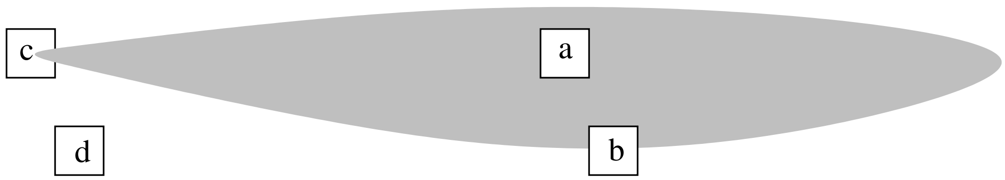


Figure 16: The gray region corresponds to $\mathcal{Z}$. Cell a is internal, cell b is mixed (with mixed east and west edges), and cells c and d are external.

for illustration.

Observation 5.2. *Consider the (infinite) graph $\mathcal{G}$ defined over the grid cells with an edge connecting every two cells that share a grid vertex (at least one). Then the subgraph induced by $\mathcal{G}$ on the collection of mixed cells is connected.*

We will soon present an iterative process, referred to as the *Boundary Reconstruction Process (BRP)*, which identifies the mixed edges, and hence also the mixed cells. The union of the mixed cells is isomorphic to a ring R such that the internal cells are enclosed by R and the external cells are outside R. This enables the classification of each point $p \in \mathbb{R}^2$ as follows:

(a) if p belongs to some internal cell C, then p is classified as a $\mathcal{Z}^+$-point and C is classified as a $\mathcal{Z}^+$-cell;

(b) else, if p belongs to some cell C at distance at most $\rho\gamma$ from some mixed cell, then it is classified as a $\mathcal{Z}^?$-point and C is classified as a $\mathcal{Z}^?$-cell;

(c) else, p is classified as a $\mathcal{Z}^-$-point.

A query on point $p \in \mathbb{R}^2$ is handled merely by computing the cell to which p belongs and deciding whether it is internal, mixed, or external, and in the latter case, deciding whether it is sufficiently close to some mixed cell. (We explain later on how this is performed in constant time.) Our analysis relies on bounding the number of mixed cells and consequently also the total number of cells that form the zone $\mathcal{Z}^?$.

The parameter γ is set to be sufficiently small so that the cell containing point s is internal. In fact, we take $\gamma \leq \min\{\tilde{\delta}^2/\tilde{\Delta}, \tilde{\delta}/(2\sqrt{2})\}$ so that (i) the ball of radius $\tilde{\delta}$ centered at s is guaranteed to contain $\Omega(\rho^2)$ cells (all of them are internal by definition); and (ii) the distance from s to any mixed cell is at least 2γ.

Recall that s is an internal vertex. Let e_1 be the unique mixed edge in the column of s to the north of s (the convexity of $\mathcal{Z}$ implies that there is indeed one such edge). Let $e_2, \ldots, e_m$ be the rest of the mixed edges of G_γ in order of discovery when traversing the closed curve $\partial\mathcal{Z}$ in clockwise direction starting from the (unique) intersection point with e_1.

BRP begins by identifying the edge e_1. This is done by applying the oracle $\mathcal{O}$ in a binary search fashion to vertices north of s at distance at most $\tilde{\Delta}$ and at least $\tilde{\delta}$ from s, so that the total number of oracle applications is $O(\log \rho)$.

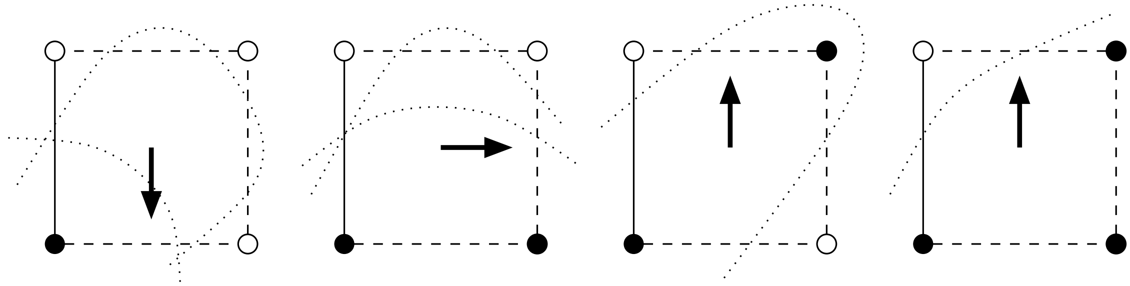


Figure 17: The four possible results of applying the oracle $\mathcal{O}$ to the south east and north east vertices of C. The mixed edge e_i (solid line) is the west edge of C. Internal vertices are depicted by full circles; external vertices are depicted by empty circles. The arrow points to the next mixed edge e_{i+1}. The dotted curves depict possible shapes of $\partial\mathcal{Z}$. By definition, these shapes "enter" C through e_i; since the south vertex of e_i is assumed to be internal, whereas its north vertex is assumed to be external, the convexity of $\mathcal{Z}$ implies that there is no need to consider shapes of $\partial\mathcal{Z}$ that "exit" C through e_i as well.

Next, the process identifies the edges $e_2, \ldots, e_m$, one by one, in an iterative manner. Suppose that the last identified mixed edge is e_i, $1 \leq i \leq m-1$, and we wish to identify the next mixed edge e_{i+1}. Assuming that $i > 1$, consider the unique grid cell having both e_i and e_{i-1} as edges and assume without loss of generality that e_i is the east (column) edge of that cell. (The other three cases are treated merely by rotating the plane.) Note that the manner in which the mixed edges are indexed (a clockwise traversal of $\partial\mathcal{Z}$) and the convexity of $\mathcal{Z}$ imply that the south vertex of e_i must be internal and the north vertex of e_i must be external; this also holds for the $i = 1$ case by the definition of e_1. Let C be the (mixed) grid cell having e_i as its west edge. The mixed edge e_{i+1} must be one of the other three edges of C. It is uniquely determined by applying the oracle $\mathcal{O}$ to the south east and north east vertices of C. Refer to Figure 17 for more details. BRP ends when the iterative process returns to the mixed edge e_1.

We now turn to the analysis. By definition, every internal cell is fully contained in $\mathcal{Z}$, thus $\mathcal{Z}^+ \subseteq \mathcal{Z}$. It remains to show that $\mathcal{Z}^- \cap \mathcal{Z} = \emptyset$ and that $\text{area}(\mathcal{Z}^?) \leq \epsilon \cdot \text{area}(\mathcal{Z})$. We start with the former.

Lemma 5.3. *If $p \in \mathbb{R}^2$ is classified as a $\mathcal{Z}^-$-point, then $p \notin \mathcal{Z}$.*

Proof. Recall that p is classified as a $\mathcal{Z}^-$-point if and only if it does not belong to an internal cell nor to any cell at distance at most $\rho\gamma$ from some mixed cell. Suppose, towards deriving contradiction, that there exists some point p which is classified as a $\mathcal{Z}^-$-point and yet $p \in \mathcal{Z}$. In this context it may be convenient to think of p and s as vectors in $\mathbb{R}^2$; as such, we denote them by $\vec{p}$ and $\vec{s}$, respectively. Let $\vec{u}$ be a unit vector orthogonal to $\vec{s} - \vec{p}$. For every real $\theta \in [0, 1]$, let $\vec{q}_\theta = \theta\vec{s} + (1-\theta)\vec{p}$ and fix

$$d(\theta) = \|\vec{p} - \vec{q}_\theta\| \quad \text{and} \quad h(\theta) = \sup\{h \in \mathbb{R}_{\geq 0} \mid \vec{q}_\theta + h\vec{u} \in \mathcal{Z} \wedge \vec{q}_\theta - h\vec{u} \in \mathcal{Z}\} \ .$$

The convexity of $\mathcal{Z}$ implies that the function $f(\theta) = h(\theta)/d(\theta)$ is non-increasing in the interval

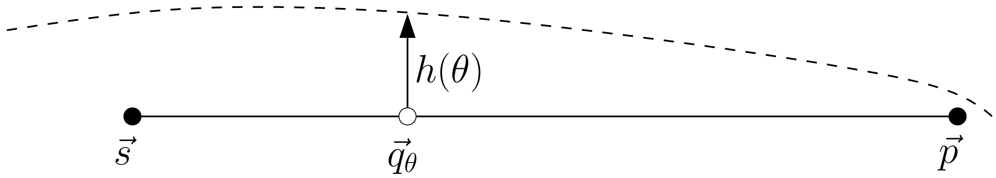


Figure 18: The point $\vec{q}_\theta$ for $\theta = 2/3$. Part of $\partial\mathcal{Z}$ is depicted by the dashed curve. The arrow represents a vector orthogonal to $\vec{s} - \vec{p}$ of magnitude $h(\theta)$.

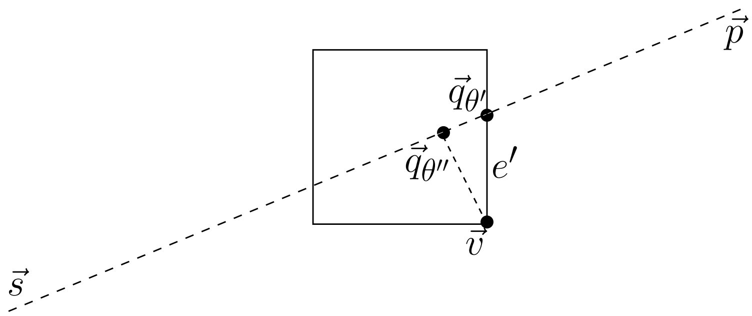


Figure 19: The mixed cell C that realizes θ' is depicted by the solid square. The vector $\vec{v} - \vec{q}_{\theta''}$ is orthogonal to $\vec{s} - \vec{p}$.

$(0, 1]$. Refer to Figure 18 for illustration.

Let $\theta' = \inf\{\theta \in (0,1) \mid \vec{q}_\theta \text{ belongs to a mixed cell}\}$. This is well defined as $\vec{p}$ belongs to an external cell and $\vec{s}$ belongs to an internal cell. Since $\vec{p}$ is a point in $\mathcal{Z}^-$ and $\vec{q}_{\theta'}$ is on an edge of a mixed cell, we have $d(\theta') > \rho\gamma$. Consider the mixed cell C that realizes θ' and let e' be the edge of C such that $\vec{q}_{\theta'} \in e'$. By the definition of θ', both vertices of e' are external as the cell preceding C when one walks from $\vec{p}$ in the direction of $\vec{s} - \vec{p}$ is not mixed. Let $\vec{v}$ be the vertex of e' that maximizes $\vec{v} \cdot (\vec{s} - \vec{p})$. Refer to Figure 19 for illustration.

Fix

$$\theta'' = \frac{(\vec{v} - \vec{p}) \cdot (\vec{s} - \vec{p})}{\|\vec{s} - \vec{p}\|^2} \quad \text{and} \quad \vec{w} = \frac{\vec{v} - \vec{q}_{\theta''}}{\|\vec{v} - \vec{q}_{\theta''}\|} \ .$$

The parameter θ'' can be thought of as the projection of $\vec{v} - \vec{p}$ on $(\vec{s} - \vec{p})/\|\vec{s} - \vec{p}\|$, normalized by a factor of $\|\vec{s} - \vec{p}\|$.

By definition, $\vec{w}$ is a unit vector. Moreover, it is orthogonal to $\vec{s} - \vec{p}$ as $\vec{w} \cdot (\vec{s} - \vec{p}) = 0$ can be equivalently expressed as $(\vec{v} - \vec{q}_{\theta''}) \cdot (\vec{s} - \vec{p}) = 0$, which holds since

$$\begin{aligned}(\vec{v} - \vec{q}_{\theta''}) \cdot (\vec{s} - \vec{p}) &= \left(\vec{v} - \frac{(\vec{v} - \vec{p}) \cdot (\vec{s} - \vec{p})}{\|\vec{s} - \vec{p}\|^2}(\vec{s} - \vec{p}) - \vec{p}\right) \cdot (\vec{s} - \vec{p}) \\ &= (\vec{v} - \vec{p}) \cdot (\vec{s} - \vec{p}) - \frac{(\vec{v} - \vec{p}) \cdot (\vec{s} - \vec{p})}{\|\vec{s} - \vec{p}\|^2}(\vec{s} - \vec{p}) \cdot (\vec{s} - \vec{p}) \\ &= (\vec{v} - \vec{p}) \cdot (\vec{s} - \vec{p}) - (\vec{v} - \vec{p}) \cdot (\vec{s} - \vec{p}) \;=\; 0\ .\end{aligned}$$

Hence, $\|\vec{q}_{\theta'} - \vec{q}_{\theta''}\| \leq \|\vec{v} - \vec{q}_{\theta'}\|$.

As $\vec{v}$ and $\vec{q}_{\theta'}$ are both on the edge e', we know that $\|\vec{q}_{\theta'} - \vec{q}_{\theta''}\| \leq \|\vec{v} - \vec{q}_{\theta'}\| \leq \gamma$. By the choice of γ, and since $\vec{q}_{\theta'}$ is on an edge of a mixed cell, we know that $\|\vec{s} - \vec{q}_{\theta'}\| > 2 \cdot \gamma$, thus $\theta'' < 1$. On the other hand, the choice of $\vec{v}$ implies that $\theta'' \geq \theta'$. It follows that $d(\theta'') \geq d(\theta') > \rho\gamma$. Since $\vec{v}$ is an external vertex, it follows that $h(\theta'') < \|\vec{q}_{\theta''} - \vec{v}\|$, and since $\|\vec{q}_{\theta''} - \vec{v}\| \leq \|\vec{q}_{\theta'} - \vec{v}\|$, we conclude that $h(\theta'') < \gamma$. Therefore $f(\theta'') = h(\theta'')/d(\theta'') < 1/\rho$.

Recall that $f(\theta) = h(\theta)/d(\theta)$ is non-increasing in the interval $(0, 1]$. Thus $f(1) \leq f(\theta'') < 1/\rho$. But as $h(1) \geq \tilde{\delta}$ and $d(1) \leq \tilde{\Delta}$, $f(1)$ must be bounded from below by $1/\rho$, yielding a contradiction. □

It remains to bound the area of $\mathcal{Z}^?$ with respect to the area of $\mathcal{Z}$. In an attempt to do so, we first bound the number (and consequently, also the total area) of mixed cells. Since each mixed edge introduces at most two mixed cells, it is sufficient to bound the number m of mixed edges. To this end, we argue that $m = O(\tilde{\Delta}/\gamma)$. Indeed, every grid column (respectively, row) that intersects with $\partial\mathcal{Z}$ introduces at most two mixed vertical (resp., horizontal) edges and there are $O(\tilde{\Delta}/\gamma)$ grid columns (resp., rows) that intersect $\partial\mathcal{Z}$. Therefore, $\mathcal{Z}^?$ contains $O(\tilde{\Delta}/\gamma)$ mixed cells.

Let $\mathcal{C}$ be the collection of external cells at distance at most $\rho\gamma$ from some mixed cell and recall that $\mathcal{Z}^?$ consists of the mixed cells and the cells in $\mathcal{C}$. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the subcollection that consists of every cell $C \in \mathcal{C}$ for which there exists some mixed cell C' such that C and C' belong to the same row or column. By Observation 5.2, $\mathcal{C}'$ can be partitioned into pairwise disjoint clusters so that each cluster is associated with a unique mixed cell and the size of each cluster is $O(\rho)$. Thus, $|\mathcal{C}'| = O(\rho\tilde{\Delta}/\gamma)$. Observing that $|\mathcal{C} - \mathcal{C}'| = O(\rho^2)$, we conclude that $|\mathcal{C}| = O(\rho\tilde{\Delta}/\gamma + \rho^2)$. As $\rho = \tilde{\Delta}/\tilde{\delta}$ and $\gamma < \tilde{\delta}$, it follows that $\mathcal{Z}^?$ contains, in total,

$$O\left(\frac{\tilde{\Delta}}{\gamma} + \frac{\rho\tilde{\Delta}}{\gamma} + \rho^2\right) = O\left(\frac{\tilde{\Delta}^2}{\gamma \cdot \tilde{\delta}}\right)$$

cells.

Since the area of each grid cell is γ^2, it follows that $\text{area}(\mathcal{Z}^?) \leq c\gamma\tilde{\Delta}^2/\tilde{\delta}$ for some constant c. In order to guarantee that $\text{area}(\mathcal{Z}^?) \leq \epsilon \cdot \text{area}(\mathcal{Z})$, we employ the fact that $\text{area}(\mathcal{Z}) \geq \pi\tilde{\delta}^2$ and demand that $c\gamma\tilde{\Delta}^2/\tilde{\delta} \leq \epsilon\pi\tilde{\delta}^2$. Therefore it is sufficient to fix

$$\gamma = \frac{\epsilon\tilde{\delta}^3}{c\tilde{\Delta}^2}\ ,$$

which means that the number of mixed edges is $m = O(\rho^3 \epsilon^{-1})$.

Each cell column in G_γ contains 0, 1, or 2 contiguous sections of $\mathcal{Z}^?$-cells. Let Q be the collection of cell columns with at least one $\mathcal{Z}^?$-cell. Clearly, $|Q| \leq m + 2\rho = O(\rho^3 \epsilon^{-1})$. Therefore the data structure `QDS` can be implemented as a vector with an entry for each cell column χ in Q that encodes the sections of $\mathcal{Z}^?$-cells in χ. The $\mathcal{Z}^+$-cells are then determined to be the cells in between two contiguous sections of $\mathcal{Z}^?$-cells. Therefore, on input point $p \in \mathbb{R}^2$, we merely have to compute the grid cell to which p belongs in order to decide in constant time whether p is a $\mathcal{Z}^+$-point, a $\mathcal{Z}^?$-point, or a $\mathcal{Z}^-$-point. Lemma 5.1 follows.

5.2 Approximate point location queries in the SINR diagram

In this section we explain the relevance of the construction presented in Section 5.1 to ϵ-approximate point location queries in the SINR diagram and establish Theorem 3. Consider some uniform power network $\mathcal{A} = \langle 2, S, \bar{1}, N, \beta, \alpha \rangle$, where $S = \{s_0, \ldots, s_{n-1}\}$ and $\alpha > 0$ and $\beta > 1$ are constants. Given some $0 \leq i \leq n-1$ and some point $p \in \mathbb{R}^2 - S$, we can clearly decide whether $p \in \mathcal{H}_i$ in time $O(n)$ by directly computing $\mathrm{SINR}_{\mathcal{A}}(s_i, p)$.

Assuming that the location of s_i is not shared by any other station (if it is, then $\mathcal{H}_i = \{s_i\}$, and point location queries pertinent to $\mathcal{H}_i$ are answered trivially), we know that s_i is an internal point of $\mathcal{H}_i$. Furthermore, Theorem 1 guarantees that the reception zone $\mathcal{H}_i$ is a convex thick zone and Theorem 4.1 provides us with a lower bound $\tilde{\delta}$ on $\delta(s_i, \mathcal{H}_i)$ and an upper bound $\tilde{\Delta}$ on $\Delta(s_i, \mathcal{H}_i)$ such that $\tilde{\Delta}/\tilde{\delta} = O\left(\sqrt[\alpha]{n}\right)$.

In fact, we can obtain much tighter bounds on $\delta(s_i, \mathcal{H}_i)$ and $\Delta(s_i, \mathcal{H}_i)$ as follows. Let r be some positive real and suppose that we are promised that $\delta(s_i, \mathcal{H}_i) < r \leq 2 \cdot \Delta(s_i, \mathcal{H}_i)$. Since Theorem 4.2 guarantees that $\delta(s_i, \mathcal{H}_i) \leq \Delta(s_i, \mathcal{H}_i) \leq O(1) \cdot \delta(s_i, \mathcal{H}_i)$, we conclude that both $\delta(s_i, \mathcal{H}_i)$ and $\Delta(s_i, \mathcal{H}_i)$ must be $\Theta(r)$, which provides a lower bound $\tilde{\delta}'$ on $\delta(s_i, \mathcal{H}_i)$ (r/c for the appropriate constant $c > 1$) and an upper bound $\tilde{\Delta}'$ on $\Delta(s_i, \mathcal{H}_i)$ ($r \cdot c$ for the appropriate constant $c > 1$) such that $\tilde{\Delta}'/\tilde{\delta}' = O(1)$.

Such a positive real r is found by directly evaluating the function $\mathrm{SINR}_{\mathcal{A}}(s_i, \cdot)$ on points to the north of s_i in an iterative fashion, starting with the point at distance $r = \tilde{\delta}$ from s_i, and doubling r in each iteration until the first point outside $\mathcal{H}_i$ is reached. The value of r at this stage clearly satisfies $\delta(s_i, \mathcal{H}_i) < r$. On the other hand, since the point checked in the previous iteration was still inside $\mathcal{H}_i$, it follows that $r/2 \leq \Delta(s_i, \mathcal{H}_i)$. Therefore, when the iterative process ends, the value of r satisfies the desired property $\delta(s_i, \mathcal{H}_i) < r \leq 2 \cdot \Delta(s_i, \mathcal{H}_i)$.

In attempt to bound the running time of this iterative process, we recall that each iteration consists of a single evaluation of the SINR function, that is, $O(n)$ time. Since in the first iteration r is set to $r = \tilde{\delta}$ and since in the last iteration r satisfies $r \leq 2\tilde{\Delta}$, it follows that the number of iterations is logarithmic in the ratio $\tilde{\Delta}/\tilde{\delta}$, which is $O(\log n)$ assuming that the path loss parameter

α is constant, thus establishing the following corollary.

Corollary 5.4. *An improved lower bound $\tilde{\delta}$ on $\delta(s_i, \mathcal{H}_i)$ and an improved upper bound $\tilde{\Delta}$ on $\Delta(s_i, \mathcal{H}_i)$ such that $\tilde{\Delta}/\tilde{\delta} = O(1)$ can be computed in time $O(n \log n)$.*

Given a performance parameter $0 < \epsilon < 1$, we apply the technique presented in Section 5.1 to $\mathcal{H}_i$, using direct computations of the function $\text{SINR}_{\mathcal{A}}(s_i, \cdot)$ as the oracle $\mathcal{O}$, with the improved bounds promised by Corollary 5.4 — recall that the ratio of these bounds is $\rho = O(1)$. Lemma 5.1 guarantees that by making $O(\rho^3 \epsilon^{-1}) = O(\epsilon^{-1})$ calls to $\mathcal{O}$, which in total require $O(n\epsilon^{-1})$ time, we can construct a data structure $\texttt{QDS}_i$ of size $O(\epsilon^{-1})$ that partitions the Euclidean plane into disjoint zones $\mathbb{R}^2 = \mathcal{H}_i^+ \cup \mathcal{H}_i^- \cup \mathcal{H}_i^?$ such that (1) $\mathcal{H}_i^+ \subseteq \mathcal{H}_i$; (2) $\mathcal{H}_i^- \cap \mathcal{H}_i = \emptyset$; and (3) $\mathcal{H}_i^?$ is bounded and its area is at most an ϵ-fraction of $\mathcal{H}_i$. Given a query point $p \in \mathbb{R}^2$, $\texttt{QDS}_i$ answers in constant time whether p is in $\mathcal{H}_i^+$, $\mathcal{H}_i^-$, or $\mathcal{H}_i^?$. We construct such a data structure $\texttt{QDS}_i$ separately for every $0 \leq i \leq n-1$. In total, this requires $n^2\epsilon^{-1}$ time and a memory of size $O(n\epsilon^{-1})$.

Recall that by Observation 2.2, point p cannot be in $\mathcal{H}_i$ unless it is closer to s_i than it is to any other station in S. Thus for such a point p there is no need to query the data structure $\texttt{QDS}_j$ for any $j \neq i$. A Voronoi diagram of linear size for the n stations is constructed in $O(n \log n)$ preprocessing time, so that given a query point $p \in \mathbb{R}^2$, we can identify the closest station s_i in time $O(\log n)$ and invoke the appropriate data structure $\texttt{QDS}_i$ (see, e.g., [9]).

Combining the Voronoi diagram with the data structures $\texttt{QDS}_i$ for all $0 \leq i \leq n-1$, we obtain a data structure $\texttt{DS}$ of size $O(n\epsilon^{-1} + n) = O(n\epsilon^{-1})$, constructed in $O(n^2\epsilon^{-1} + n \log n) = O(n^2\epsilon^{-1})$ preprocessing time, that given a query point $p \in \mathbb{R}^2$,
(1) finds in time $O(\log n)$ the station s_i closest to p (note that if there is no such unique station s_i, then the assumption that $n \geq 3$ implies that no station is heard at p); and
(2) decides in time $O(1)$ whether the point p is in $\mathcal{H}_i^+$, $\mathcal{H}_i^?$, or neither which means that $p \in \mathcal{H}^- = \bigcap_{i=0}^{n-1} \mathcal{H}_i^-$.
Theorem 3 follows.

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Appendix: Additional Proofs

Proof of Lemma 2.4. Fix some $x \in (0,1)$ and let

$$F(y,z) = \sigma \left(\frac{x}{y}\right)^{\alpha} + \tau \left(\frac{x}{z}\right)^{\alpha} \quad \text{and} \quad G(y,z) = \sigma \left(\frac{1-x}{1-y}\right)^{\alpha} + \tau \left(\frac{1-x}{1-z}\right)^{\alpha} \ .$$

It is easy to verify that the function $F(y,z)$ (respectively, $G(y,z)$) is continuous and strictly decreasing (resp., increasing) in both variables y and z. Therefore, $\max\{F(y,z), G(y,z)\}$ is minimized when $F(y,z) = G(y,z)$ and letting $H(y,z) = F(y,z) - G(y,z)$, it suffices to show that the solution to the following optimization problem is at least $\sigma + \tau$:

$$\begin{aligned} &\min \{F(y,z)\} \\ &\text{s.t. } H(y,z) = 0 \\ &\quad\ y, z \in (0,1) \ . \end{aligned}$$

We begin by understanding the structure of the contour

$$H_0 = \{(y,z) \in (0,1)^2 \mid H(y,z) = 0\} \ .$$

By definition, the point (y,z) satisfies $H(y,z) = 0$ if and only if

$$\left(\frac{x}{z}\right)^{\alpha} - \left(\frac{1-x}{1-z}\right)^{\alpha} = \frac{\sigma}{\tau} \left[\left(\frac{1-x}{1-y}\right)^{\alpha} - \left(\frac{x}{y}\right)^{\alpha} \right] \ .$$

Fix some $y \in (0,1)$. Since the function $f(z) = \left(\frac{x}{z}\right)^{\alpha} - \left(\frac{1-x}{1-z}\right)^{\alpha}$ is continuous and strictly decreasing in $z \in (0,1)$ with $\lim_{z \to 0+} f(z) = +\infty$ and $\lim_{z \to 1-} f(z) = -\infty$, it follows that there exists a unique $z \in (0,1)$ such that $f(z) = \frac{\sigma}{\tau} \left[\left(\frac{1-x}{1-y}\right)^{\alpha} - \left(\frac{x}{y}\right)^{\alpha} \right]$, that is, a unique $z \in (0,1)$ that satisfies $H(y,z) = 0$ for the fixed value of y. Alternatively, fix some $z \in (0,1)$. Since the function $g(y) = \frac{\sigma}{\tau} \left[\left(\frac{1-x}{1-y}\right)^{\alpha} - \left(\frac{x}{y}\right)^{\alpha} \right]$ is continuous and strictly increasing in $y \in (0,1)$ with $\lim_{y \to 0+} = -\infty$ and $\lim_{y \to 1-} = +\infty$, it follows that there exists a unique $y \in (0,1)$ such that $g(y) = \left(\frac{x}{z}\right)^{\alpha} - \left(\frac{1-x}{1-z}\right)^{\alpha}$, that is, a unique $y \in (0,1)$ that satisfies $H(y,z) = 0$ for the fixed value of z. Therefore, there exists a continuous, strictly decreasing, surjective function $\mu : (0,1) \to (0,1)$ so that the contour H_0 is depicted by the curve $\{(y, \mu(y)) \mid y \in (0,1)\}$.

We do not attempt to fully understand the function μ, however, we do notice that $\lim_{y \to 0+} \mu(y) = 1$ and that $\lim_{y \to 1-} \mu(y) = 0$. In particular, this means that $\lim_{y \to 0+} F(y, \mu(y)) = \lim_{y \to 1-} F(y, \mu(y)) = +\infty$. Therefore, the minimum of $F(y, \mu(y))$ exists and is attained at some internal point $y \in (0,1)$. It follows that the minimum of $F(y,z)$ along the curve H_0 is attained at some point $(y,z) \in H_0$ such that the gradient $\vec{\nabla} F(y,z) = \left(\frac{\partial F}{\partial y}(y,z), \frac{\partial F}{\partial z}(y,z)\right)$ of F at (y,z) is collinear[6] with the gradient $\vec{\nabla} H(y,z) = \left(\frac{\partial H}{\partial y}(y,z), \frac{\partial H}{\partial z}(y,z)\right)$ of H at (y,z). (The last claim

[6] In this context, two vectors are said to be collinear if one is a non-zero scalar multiplication of the other.

is a basic fact in multivariate calculus that is very useful in mathematical optimization and in particular, constitutes the main principle behind the Lagrange multipliers method; see, e.g., [5].)

It is easy to verify that the point $(y = x, z = x)$ belongs to the curve H_0. We argue that the minimum of $F(y, z)$ along the curve H_0 is attained at that point. This establishes the assertion as $F(x, x) = \sigma + \tau$.

To that end, observe that

$$\begin{aligned} \frac{\partial F}{\partial y}(y,z) &= -\alpha\sigma\frac{x^\alpha}{y^{\alpha+1}} , \\ \frac{\partial F}{\partial z}(y,z) &= -\alpha\tau\frac{x^\alpha}{z^{\alpha+1}} , \\ \frac{\partial H}{\partial y}(y,z) &= -\alpha\sigma\left(\frac{x^\alpha}{y^{\alpha+1}} + \frac{(1-x)^\alpha}{(1-y)^{\alpha+1}}\right) , \text{ and} \\ \frac{\partial H}{\partial z}(y,z) &= -\alpha\tau\left(\frac{x^\alpha}{z^{\alpha+1}} + \frac{(1-x)^\alpha}{(1-z)^{\alpha+1}}\right) . \end{aligned} \tag{6}$$

Since these partial derivatives are non-zero for every $(y, z) \in (0, 1) \times (0, 1)$, we conclude that for any point $(y, z) \in (0, 1) \times (0, 1)$ (not just points in H_0), $\vec{\nabla} F(y, z)$ is collinear with $\vec{\nabla} H(y, z)$ if and only if

$$\frac{\frac{\partial F}{\partial y}(y,z)}{\frac{\partial F}{\partial z}(y,z)} = \frac{\frac{\partial H}{\partial y}(y,z)}{\frac{\partial H}{\partial z}(y,z)} . \tag{7}$$

By (6), this condition can be written as

$$\frac{\sigma}{\tau}\cdot\left(\frac{z}{y}\right)^{\alpha+1} = \frac{\sigma}{\tau}\cdot\frac{\frac{x^\alpha}{y^{\alpha+1}} + \frac{(1-x)^\alpha}{(1-y)^{\alpha+1}}}{\frac{x^\alpha}{z^{\alpha+1}} + \frac{(1-x)^\alpha}{(1-z)^{\alpha+1}}} ,$$

or equivalently as

$$x^\alpha + (1-x)^\alpha\left(\frac{z}{1-z}\right)^{\alpha+1} = x^\alpha + (1-x)^\alpha\left(\frac{y}{1-y}\right)^{\alpha+1} ,$$

or, after further simplification, as

$$\frac{z}{1-z} = \frac{y}{1-y} ,$$

which, in turn, holds if and only if $y = z$.

It remains to show that along the curve H_0, (7) holds if and only if $x = y = z$. Indeed, the line $z = y$ is strictly increasing (with respect to y), thus it intersects the strictly decreasing curve $H_0 = \{(y, \mu(y)) \mid y \in (0, 1)\}$ in exactly one point: $(y = x, z = x)$. The assertion follows. □

Proof of Observation 2.5. Consider some reals $a \geq b > 1$, and $\alpha > 0$. We prove that the first derivative of $\frac{\sqrt[\alpha]{a+c}+1}{\sqrt[\alpha]{b+c}-1}$ with respect to c is negative in $[0, \infty)$, which yields the observation. Note that, $\frac{\partial \sqrt[\alpha]{x+c}}{\partial c} = \frac{\sqrt[\alpha]{x+c}}{\alpha(x+c)}$. We have,

$$\begin{aligned}
\frac{\partial(\sqrt[\alpha]{a+c}+1)/(\sqrt[\alpha]{b+c}-1)}{\partial c} &= \frac{(\sqrt[\alpha]{b+c}-1)\cdot\frac{\sqrt[\alpha]{a+c}}{\alpha(a+c)} - (\sqrt[\alpha]{a+c}+1)\cdot\frac{\sqrt[\alpha]{b+c}}{\alpha(b+c)}}{(\sqrt[\alpha]{b+c}-1)^2} \\
&\leq \frac{(\sqrt[\alpha]{b+c}-1)\cdot\sqrt[\alpha]{a+c} - (\sqrt[\alpha]{a+c}+1)\cdot\sqrt[\alpha]{b+c}}{\alpha(b+c)(\sqrt[\alpha]{b+c}-1)^2} \\
&= \frac{-\sqrt[\alpha]{a+c} - \sqrt[\alpha]{b+c}}{\alpha(b+c)(\sqrt[\alpha]{b+c}-1)^2} < 0,
\end{aligned}$$

where the second inequality holds, since $b \leq a$ and $\sqrt[\alpha]{b+c} - 1 \geq 0$. The observation follows. □