Proportional Representation under Single-Crossing Preferences Revisited

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1.

Framework

Multiwinner Voting &
The Chamberlin-Courant Rule

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(preference profile)

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Q: How do we pick the K-committee?

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V1 :	Blue (0)	>	Yellow (1)	>	Red (5)	>	Pink (8)	>	Green (9)
V2 :	Yellow (0)	>	Green (3)	>	Red (3)	>	Pink (4)	>	Blue (8)
V3 :	Green (0)	>	Red (1)	>	Blue (1)	>	Pink (2)	>	Yellow (3)

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```
                1                8
V1 :       >  Yellow  >       >  Pink  >
       0                         4
V2 : Yellow  >        >       >  Pink  >
                                 2         3
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Hardness of CC

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Utilitarian-CC is NP-hard

[Procaccia, Rosenschein, Zohar'08]

[Lu, Boutilier'11]

Egalitarian-CC is NP-hard

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Real elections have more structure, making CC easier! We consider ***single-crossing*** preferences.

[Roberts'77, Mirrlees'71]

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Structured Preferences

Single-crossing Preferences & Intermediate Preferences on Median Graphs

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Difficulty: Preserve Condorcet domain and poly-time solvability of CC.

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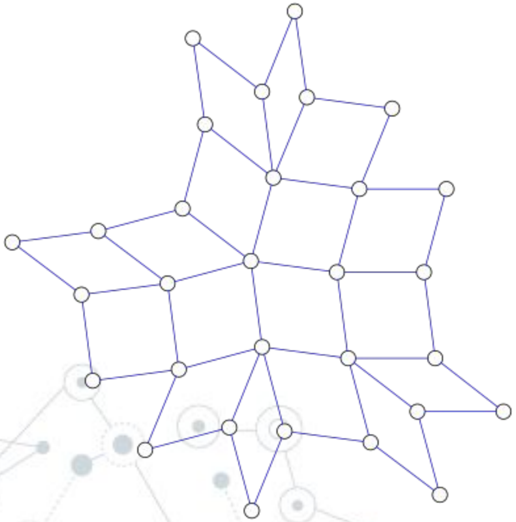

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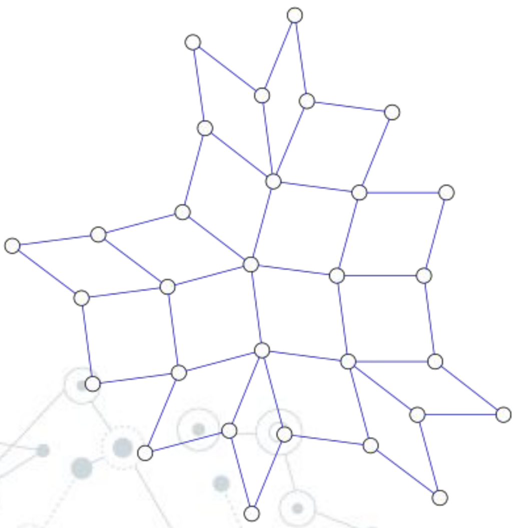

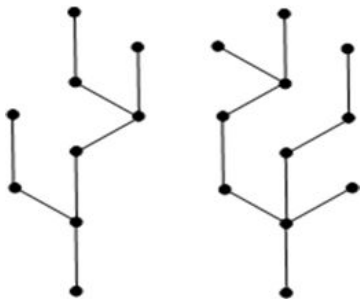

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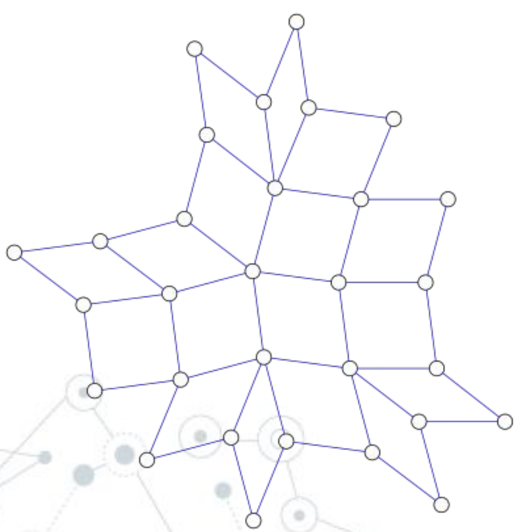

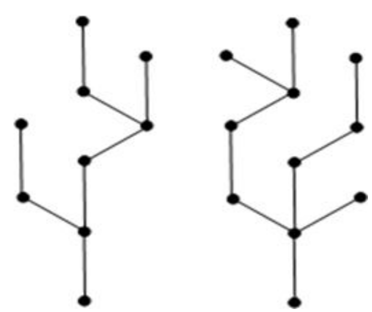

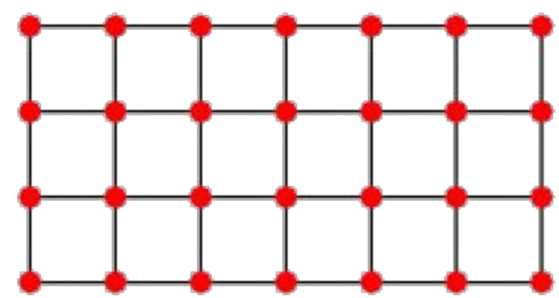

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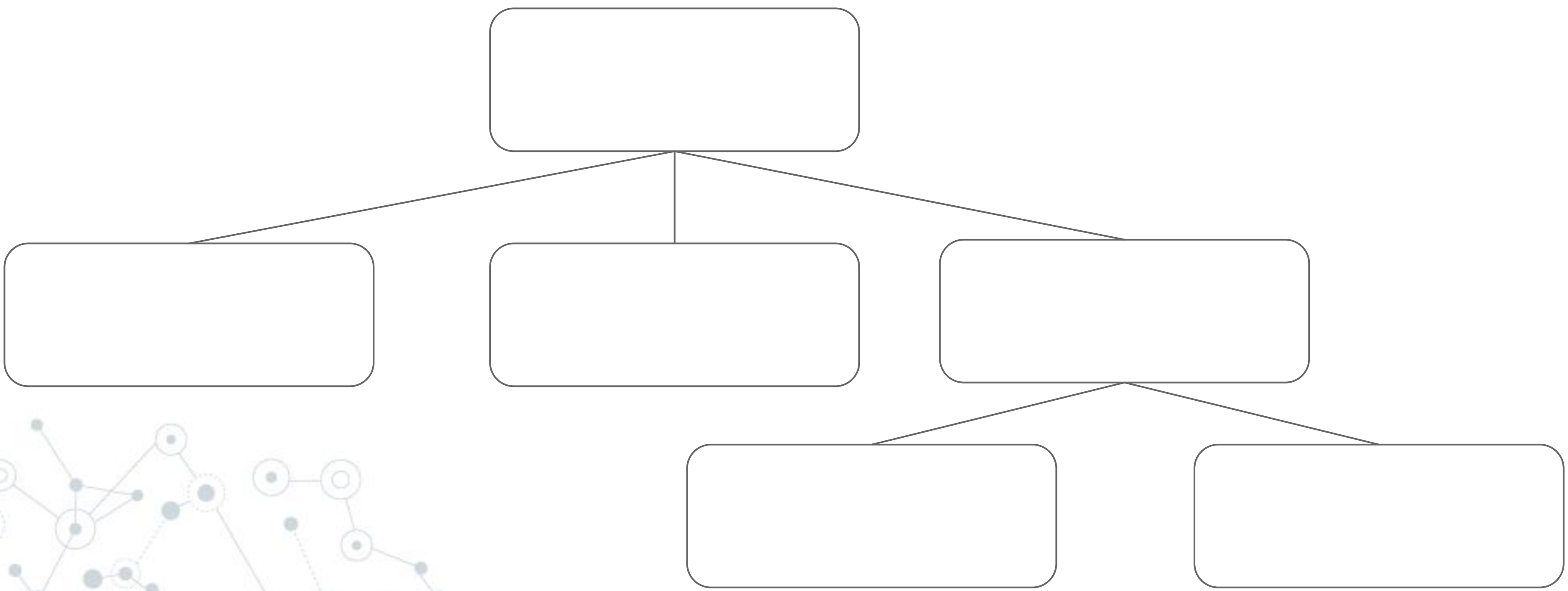

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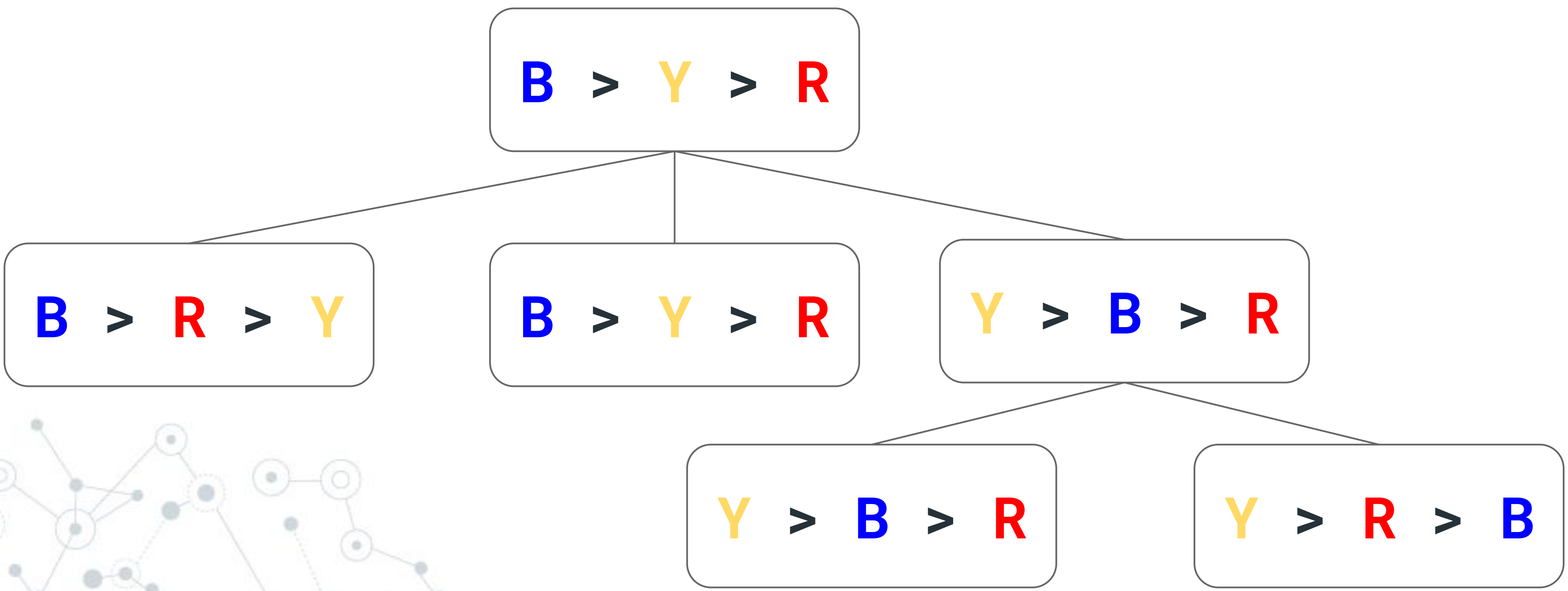

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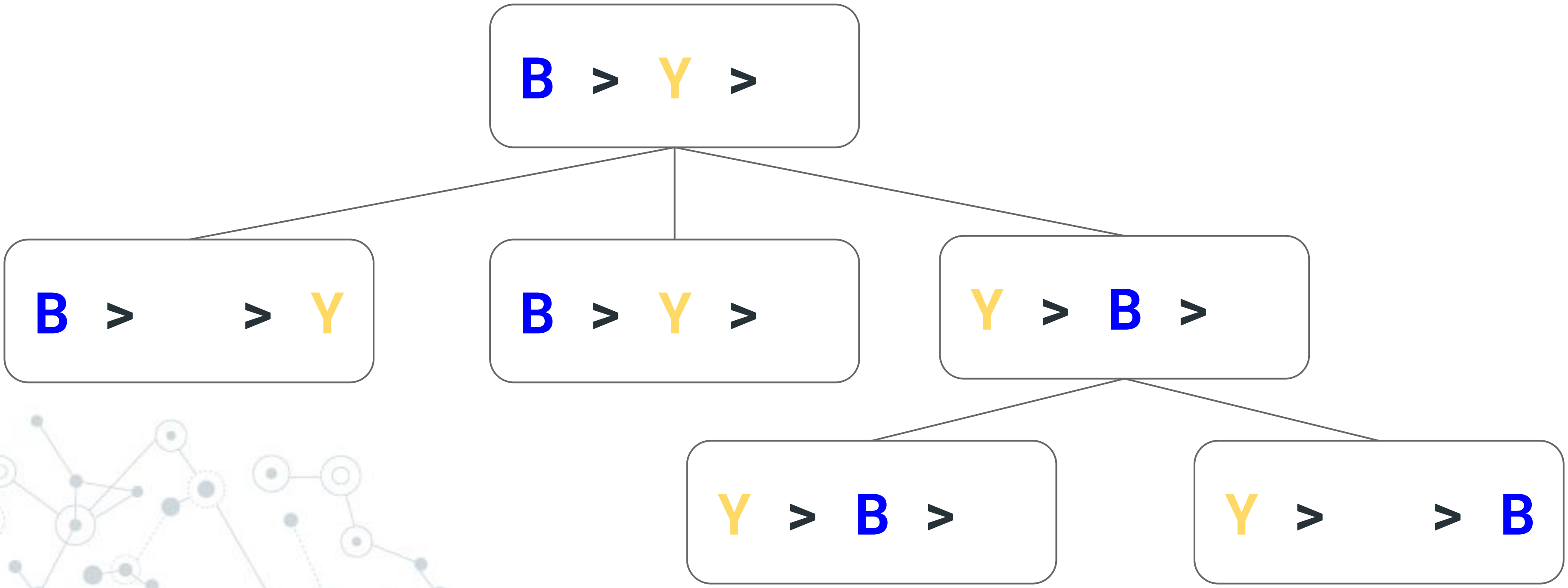

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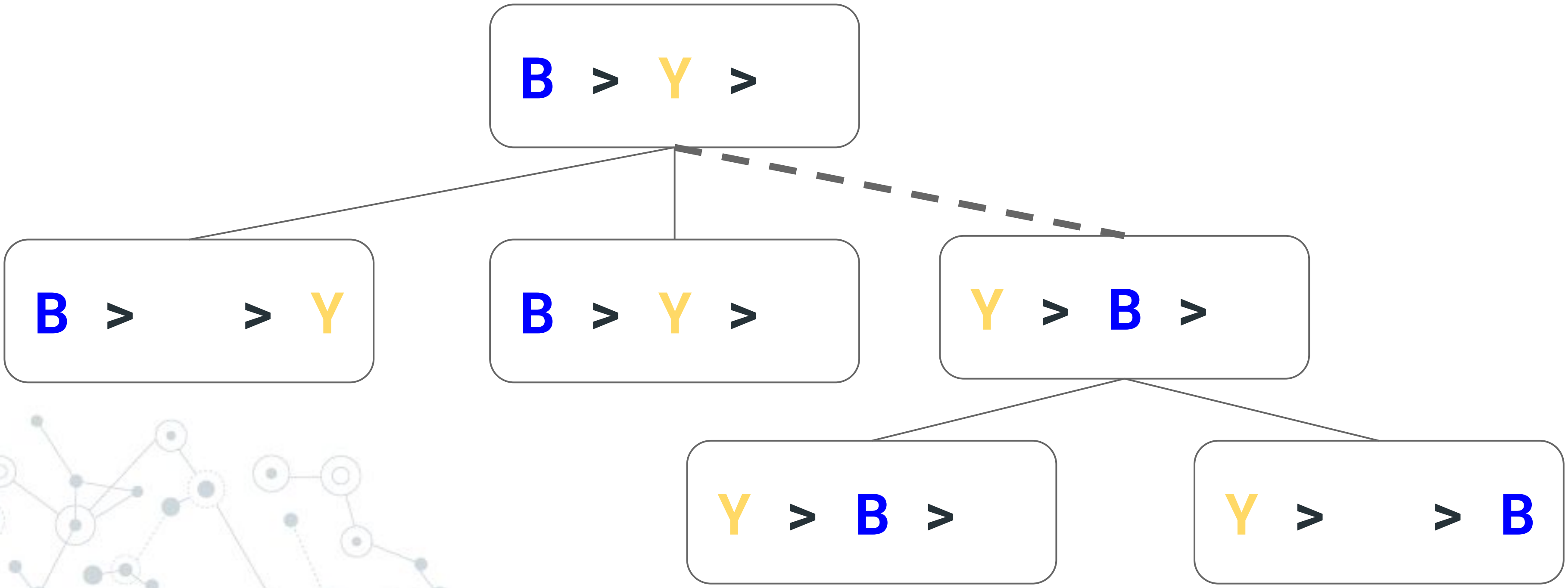

3.

This Paper

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3. **Not in this talk**: Conjecture DP algorithm for CC under grid-SC.

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Blue	>		>	Red	>	Green
Blue	>	Red	>		>	Green
Red	>	Blue	>	Green	>	
Red	>	Green	>		>	Blue
Green	>	Red	>		>	Blue

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This allows simple interval DP to work [Skowron et al.'15], with more care it can be implemented in $O(nmk)$.

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Green	>	Red	>		>	Blue

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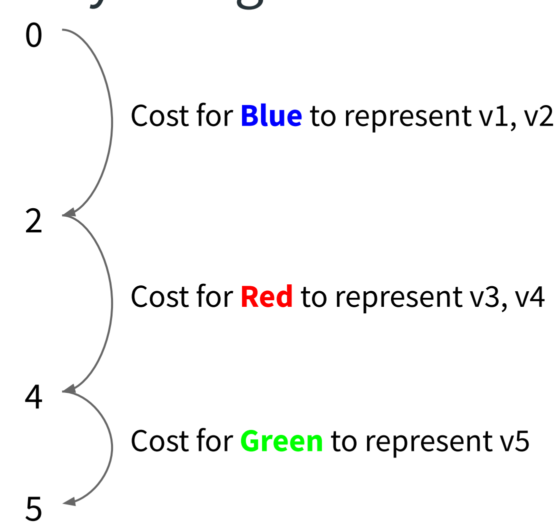

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Remark For egalitarian, binary search the answer and then run algorithm on instance with 0-1 dissatisfactions. This gives *O(nm log n log (nm))*.

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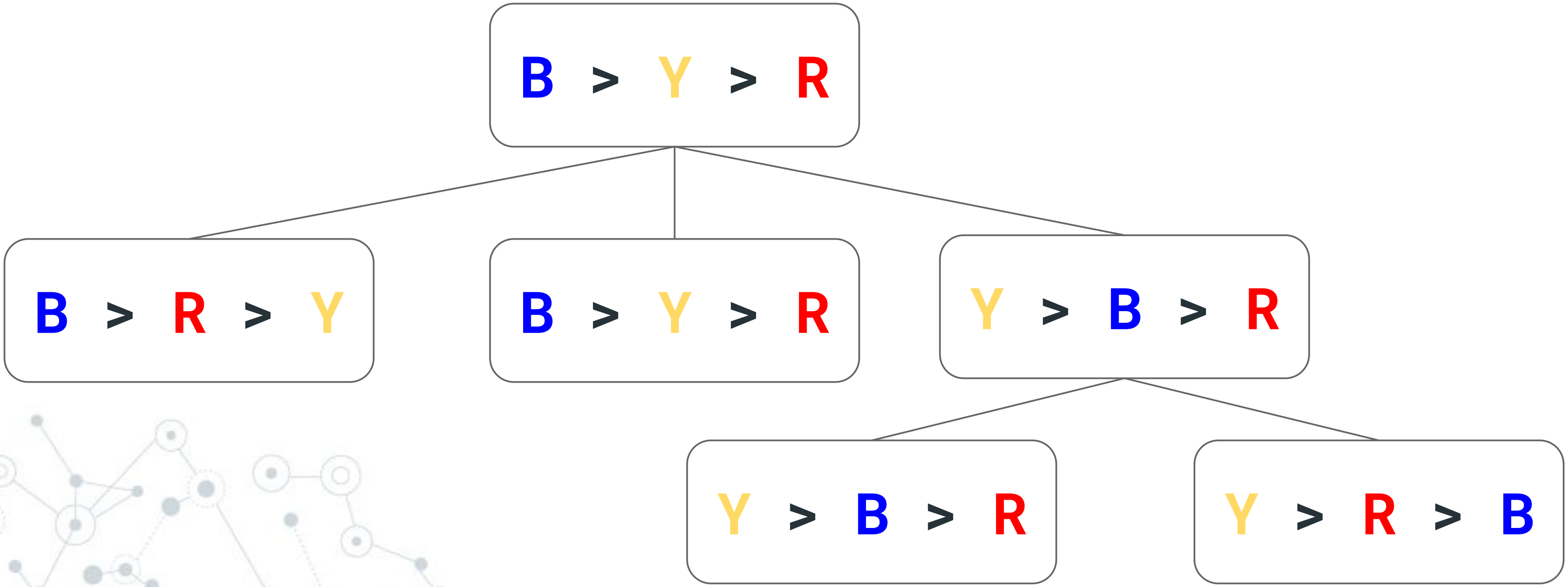

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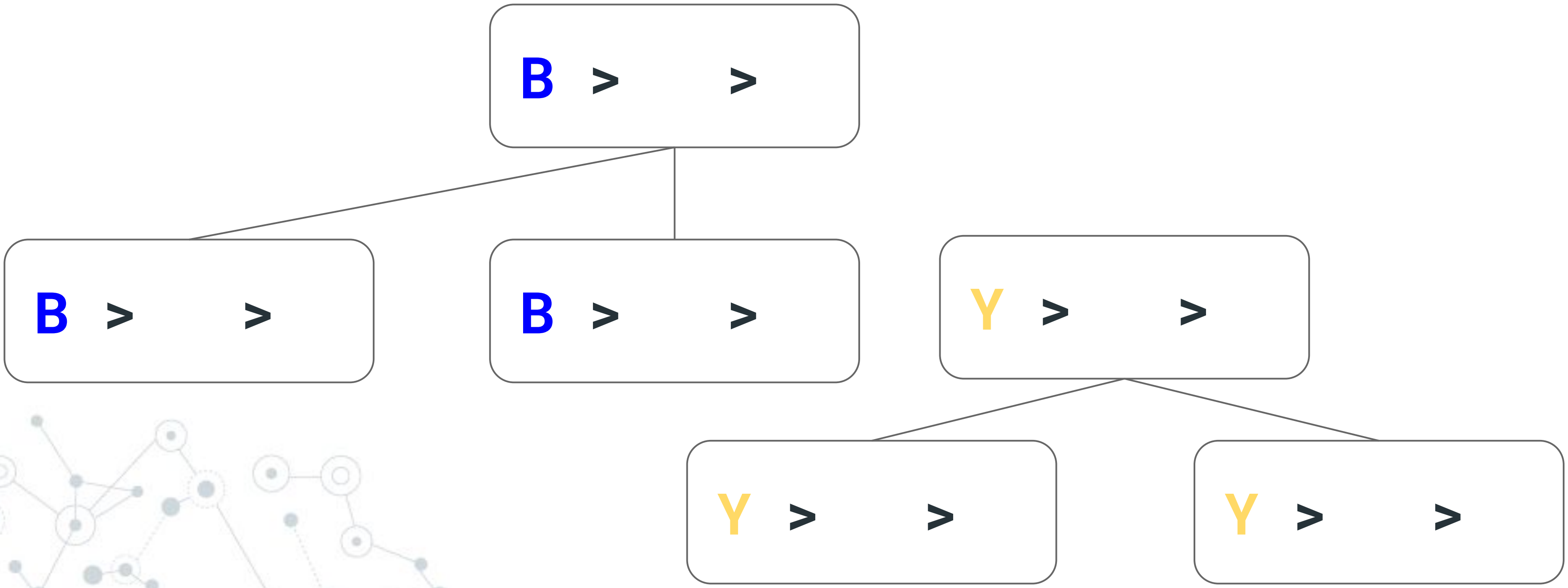

CC Under Tree-SC

Assume candidates are numbered 1, 2, …, M. Root the tree and assume that the root has the order 1 > 2 > … > M.

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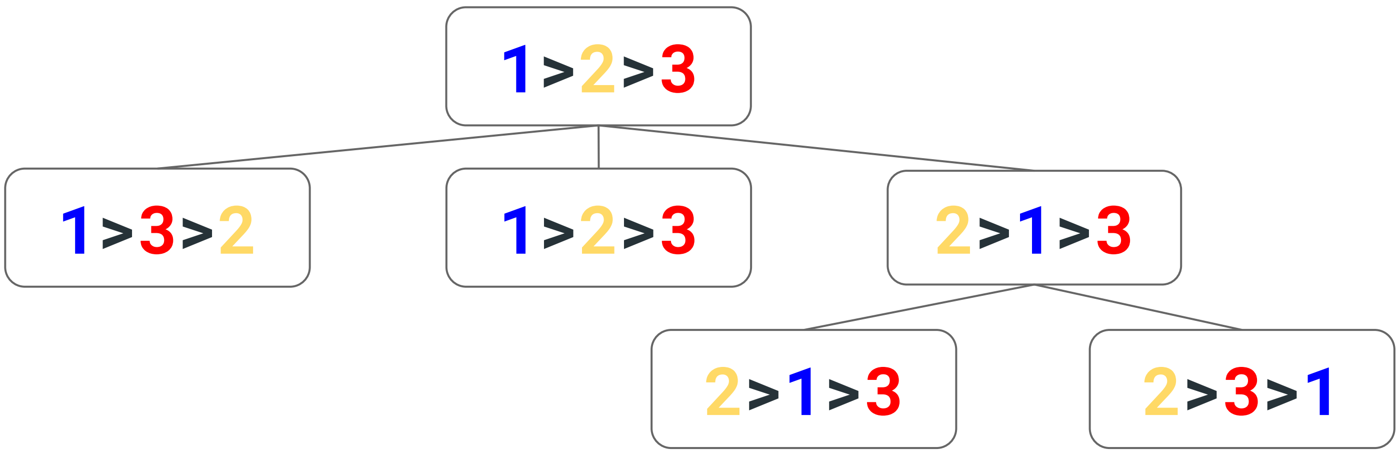

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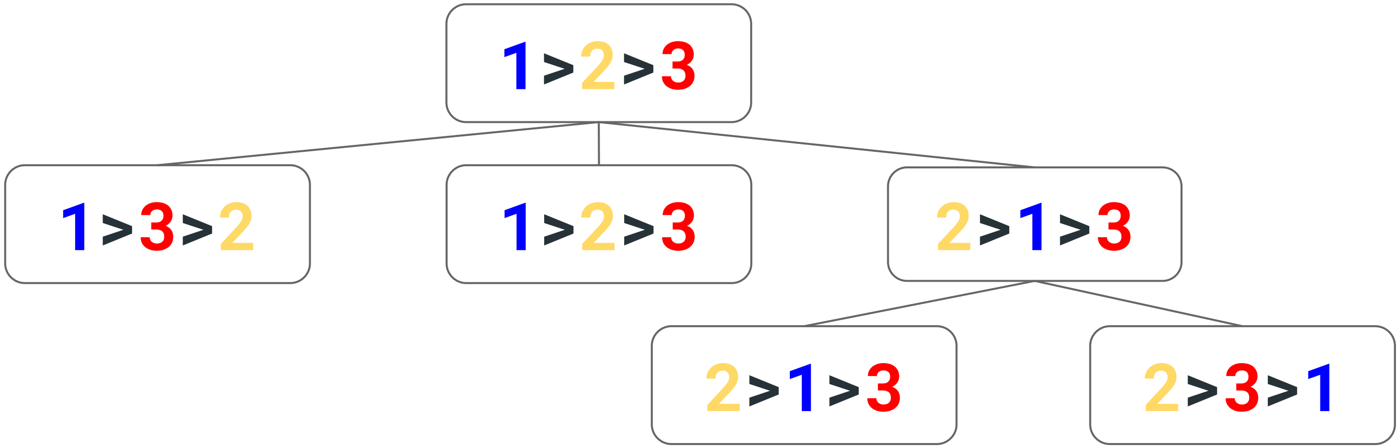

Monotonicity Lemma

In any K-committee, while walking down the tree the representing candidate is non-decreasing.

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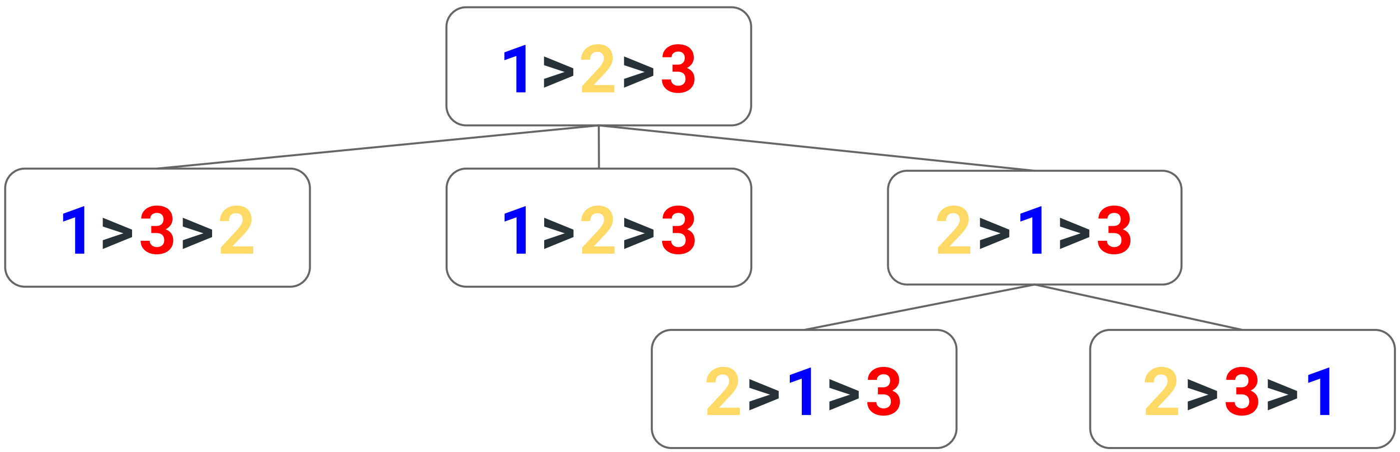

Monotonicity Lemma (ins. [Clearwater et al.'15])

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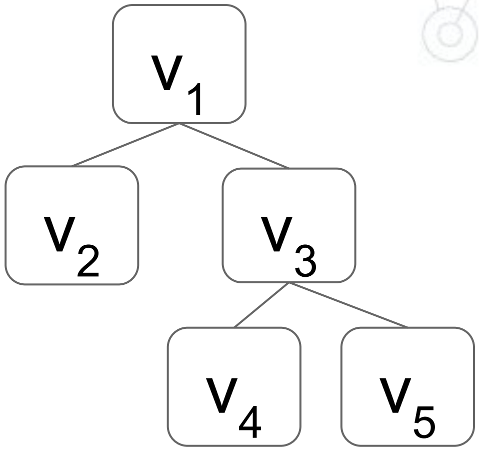

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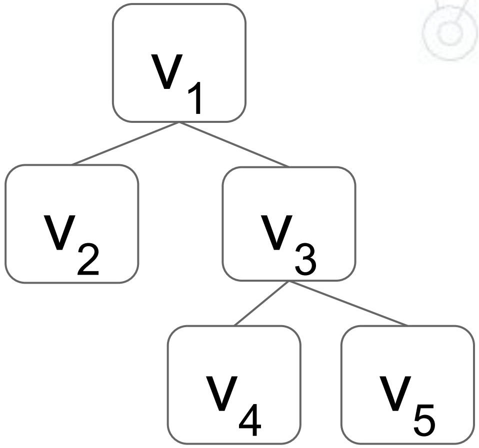

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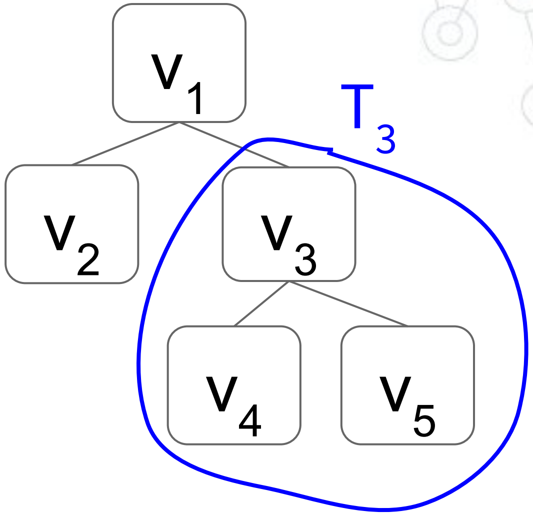

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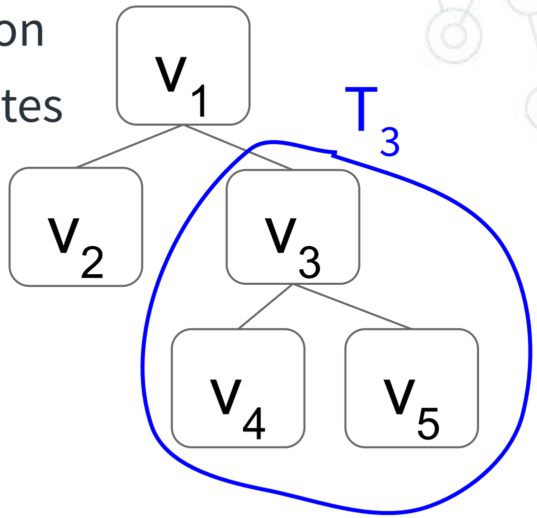

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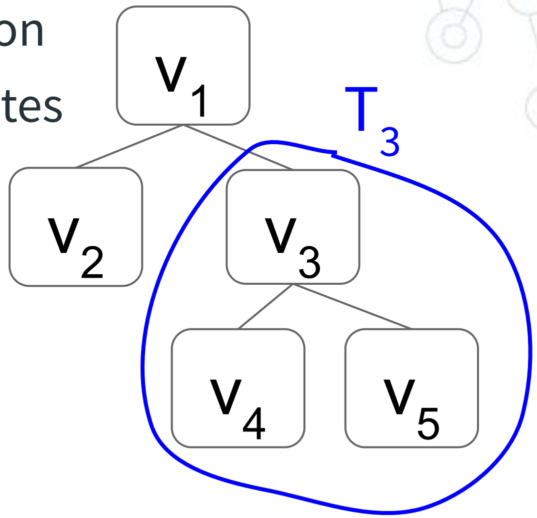

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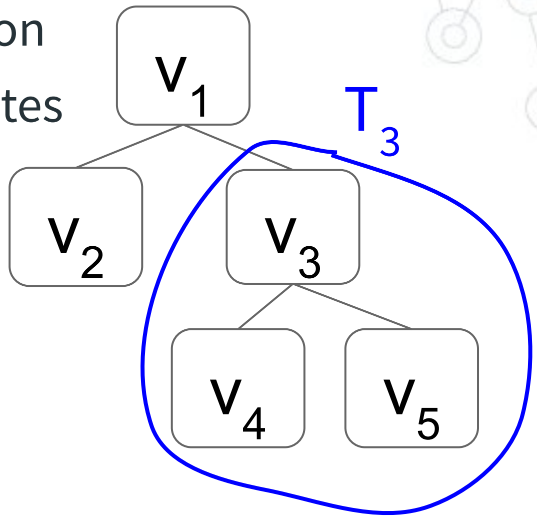

Interesting case: A node v with two children l and r.

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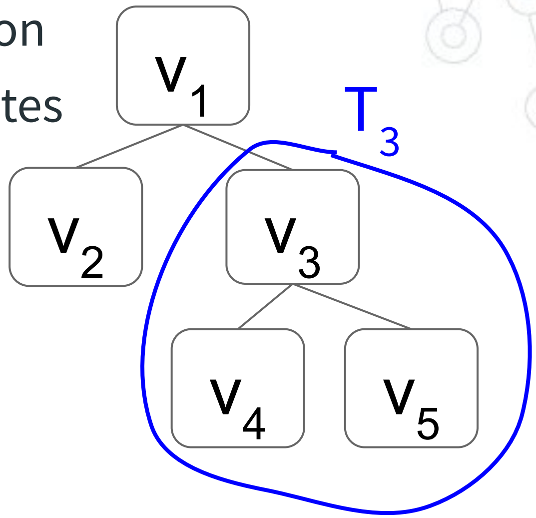

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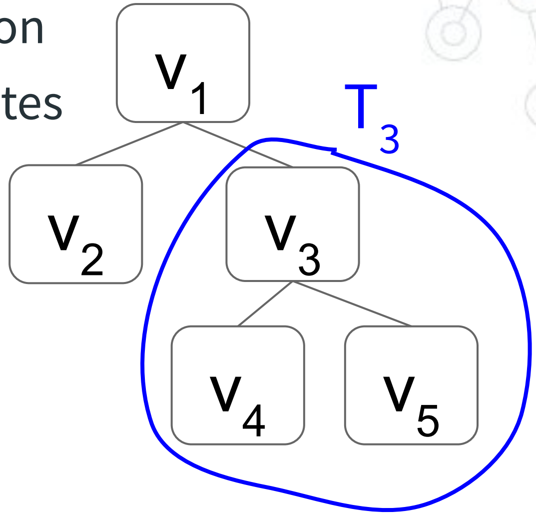

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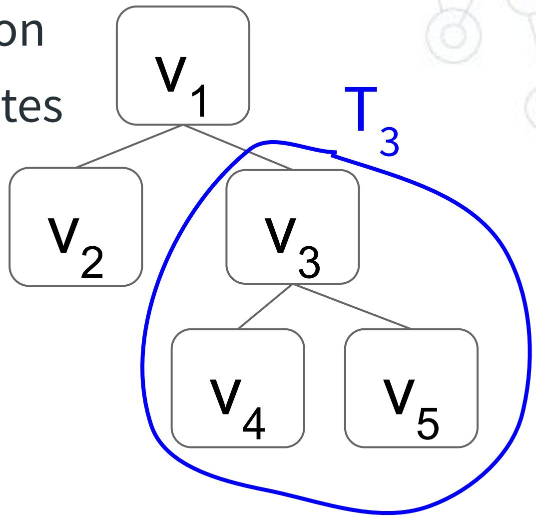

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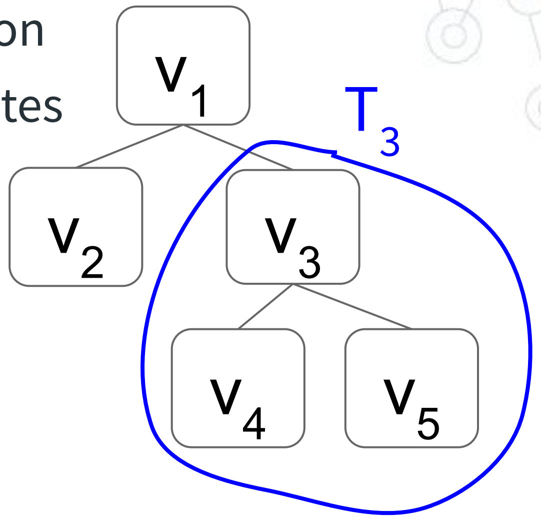

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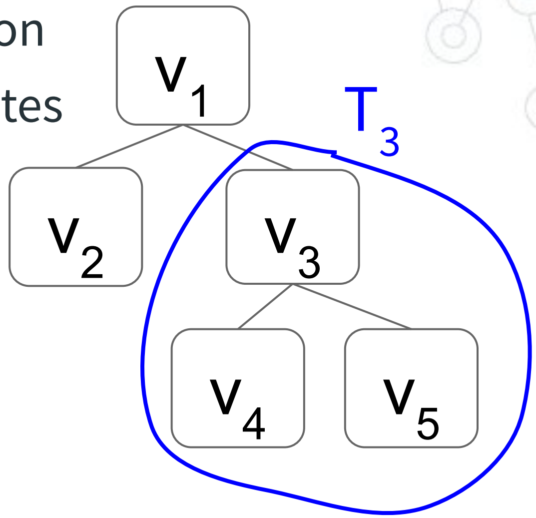

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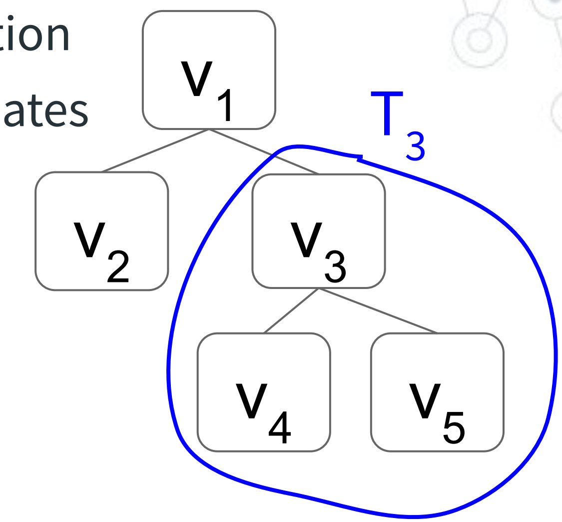

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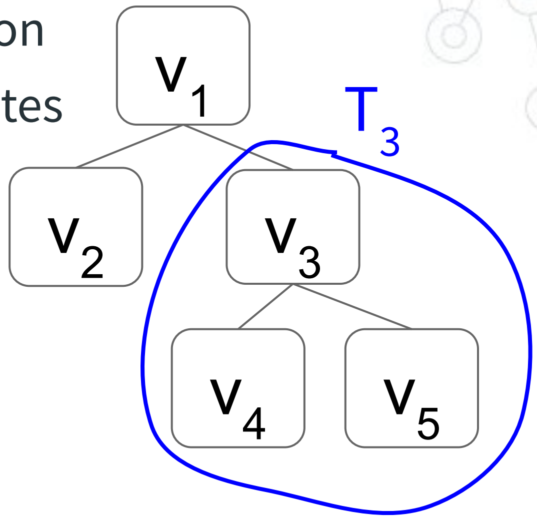

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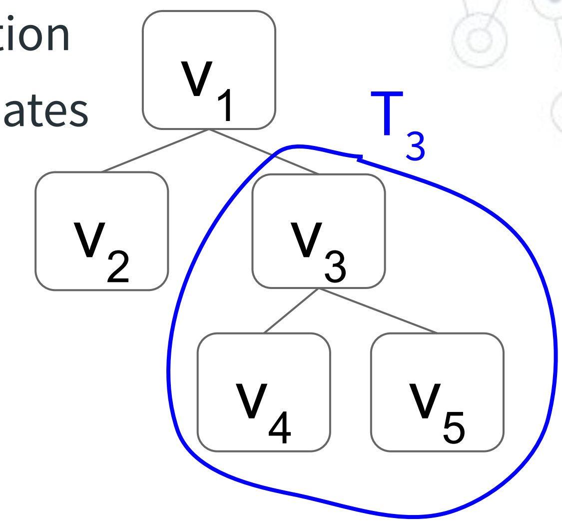

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- With care can be implemented in *O(nmk)*.

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Hope you enjoyed!

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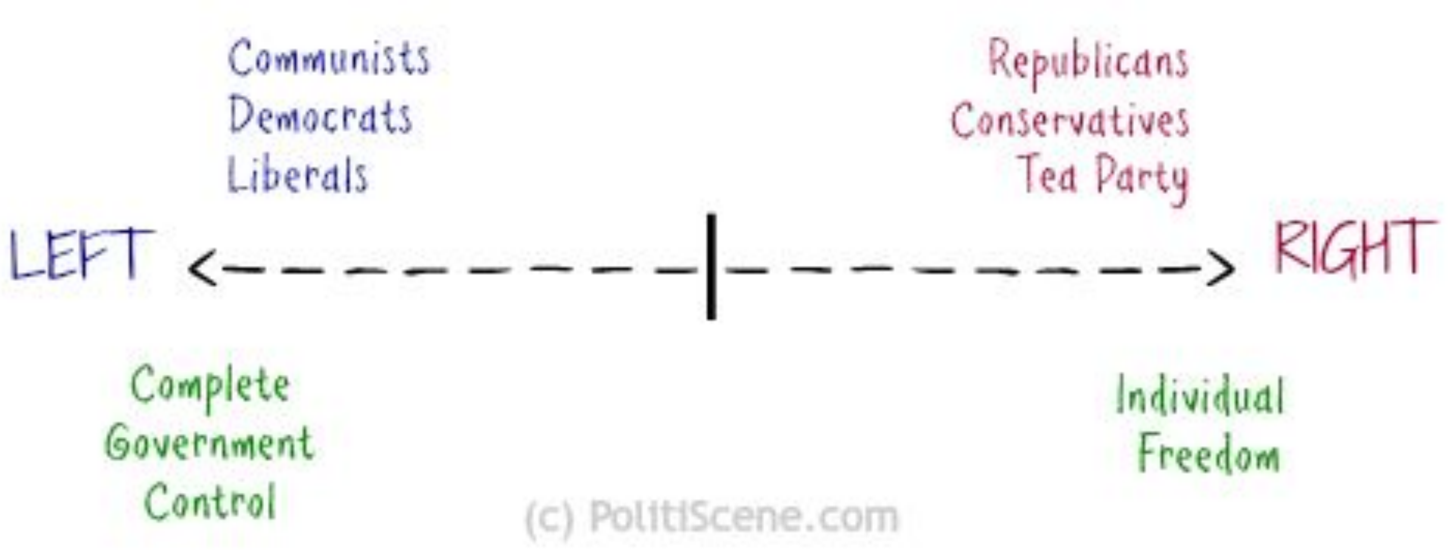

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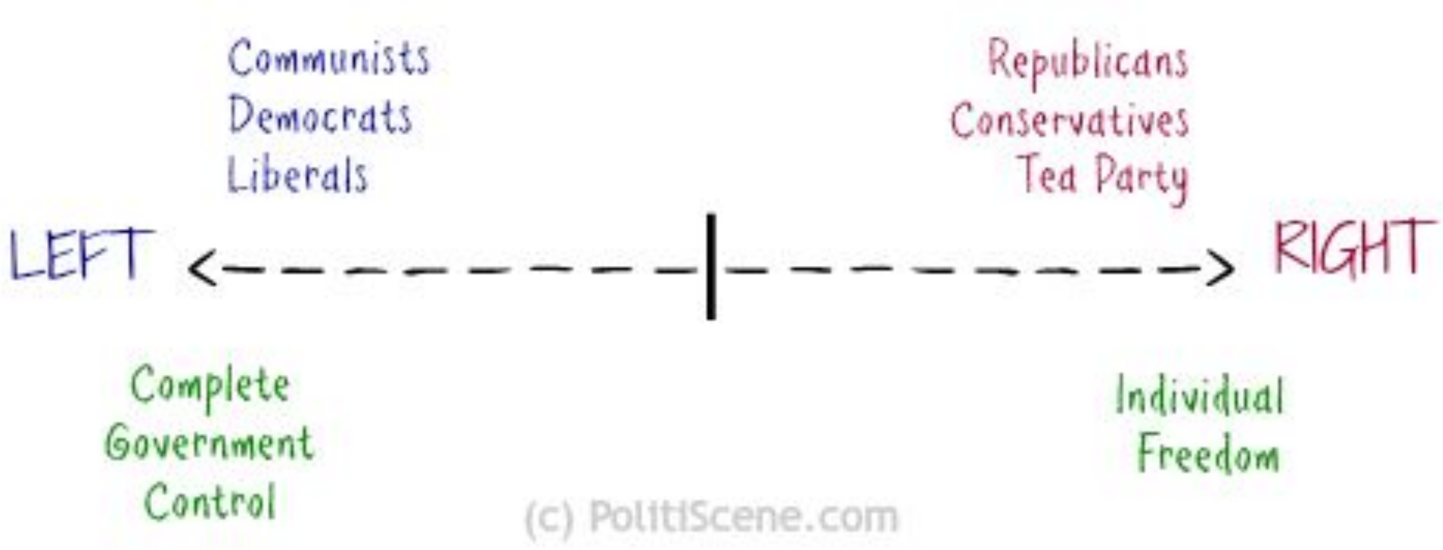

Voters vote based on how far off a candidate's number is from their own.