A Lower Bound for the Distributed Lovász Local Lemma

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The Lovász Local Lemma

- «Bad» events $E_1, E_2, \ldots, E_n$ with $\Pr(E_i) < 1$

mutually independent

$$\Rightarrow \Pr(\neg E_1 \wedge \neg E_2 \wedge \cdots \wedge \neg E_n) > 0$$

Lovász Local Lemma

- Each event is independent of all but d other events
- $\Pr(E_i) < p$ where $ep(d+1) \leq 1$

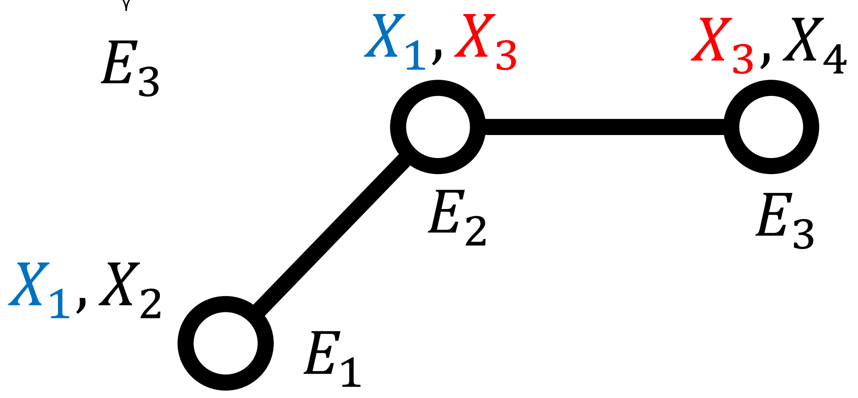
$$\Rightarrow \Pr(\neg E_1 \wedge \neg E_2 \wedge \cdots \wedge \neg E_n) > 0$$

The Constructive LLL

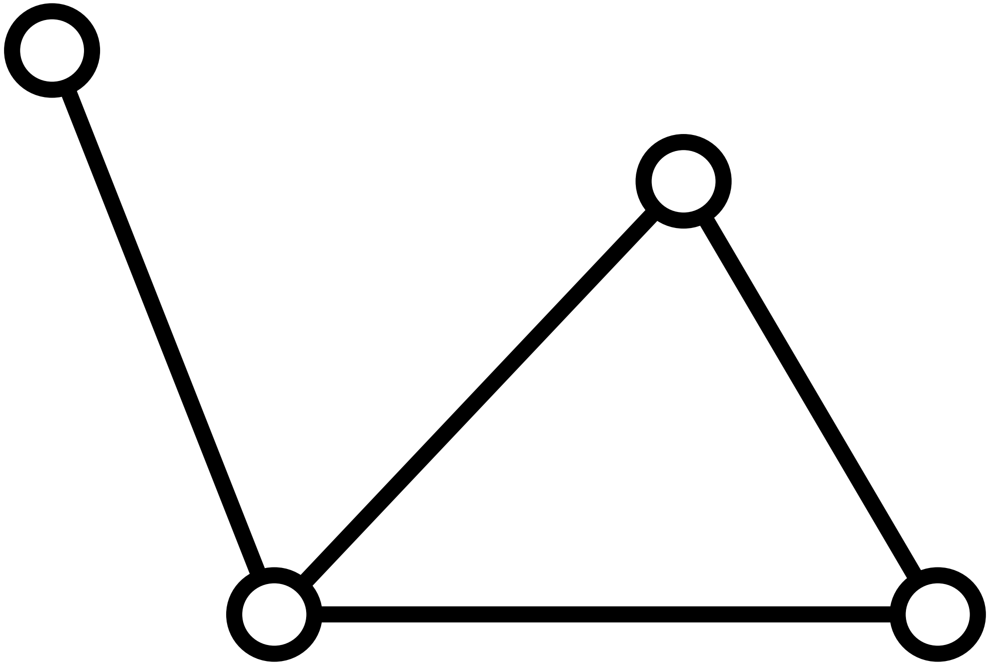
- Mutually independent random variables $X_1, X_2, \ldots, X_k$
- Bad events $E_1, E_2, \ldots, E_n$
- Each event is independent of all but d other events

$$\underbrace{(X_1 \lor \neg X_2)}_{E_1} \land \underbrace{(\neg X_1 \lor X_3)}_{E_2} \land \underbrace{(X_3 \lor X_4)}_{E_3}$$

- Dependency graph



Distributed Computing



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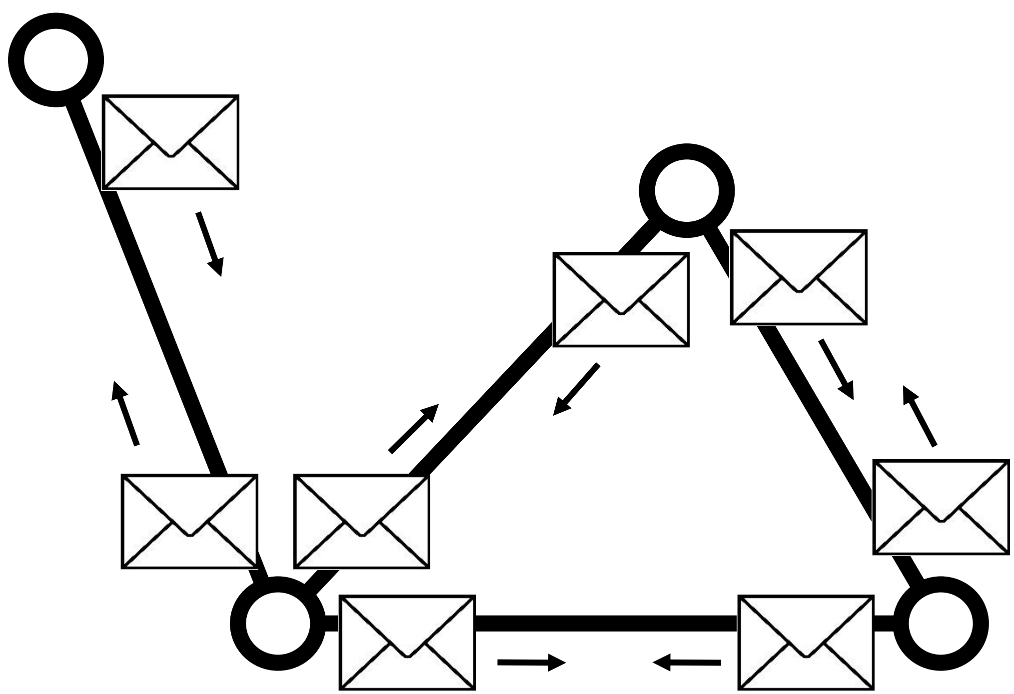

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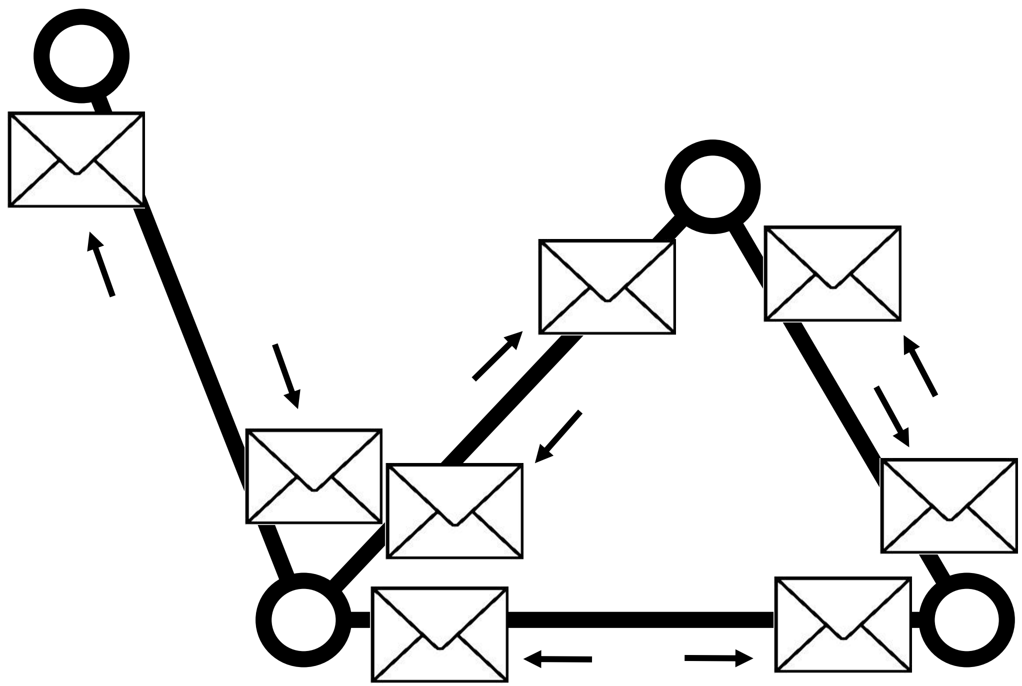

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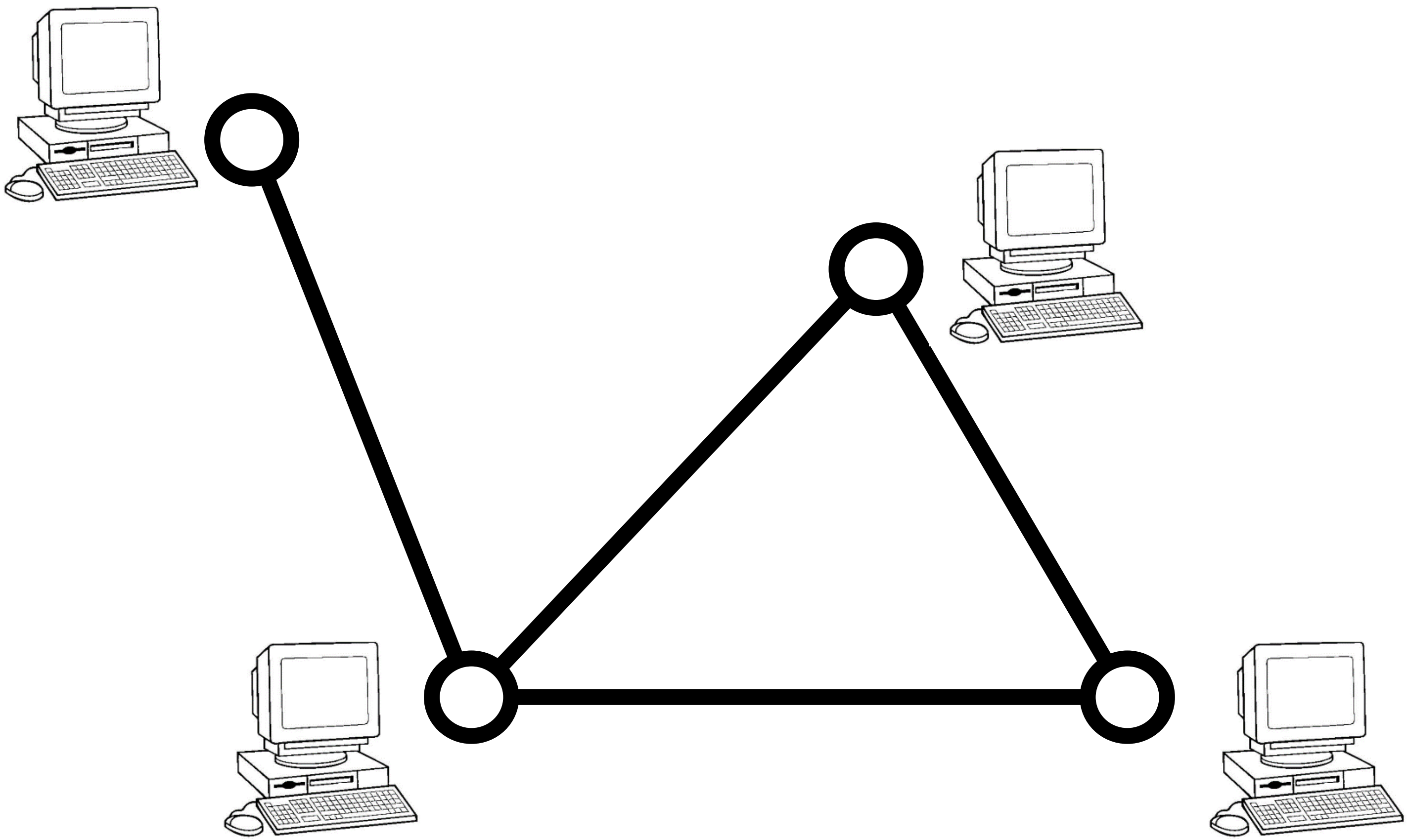

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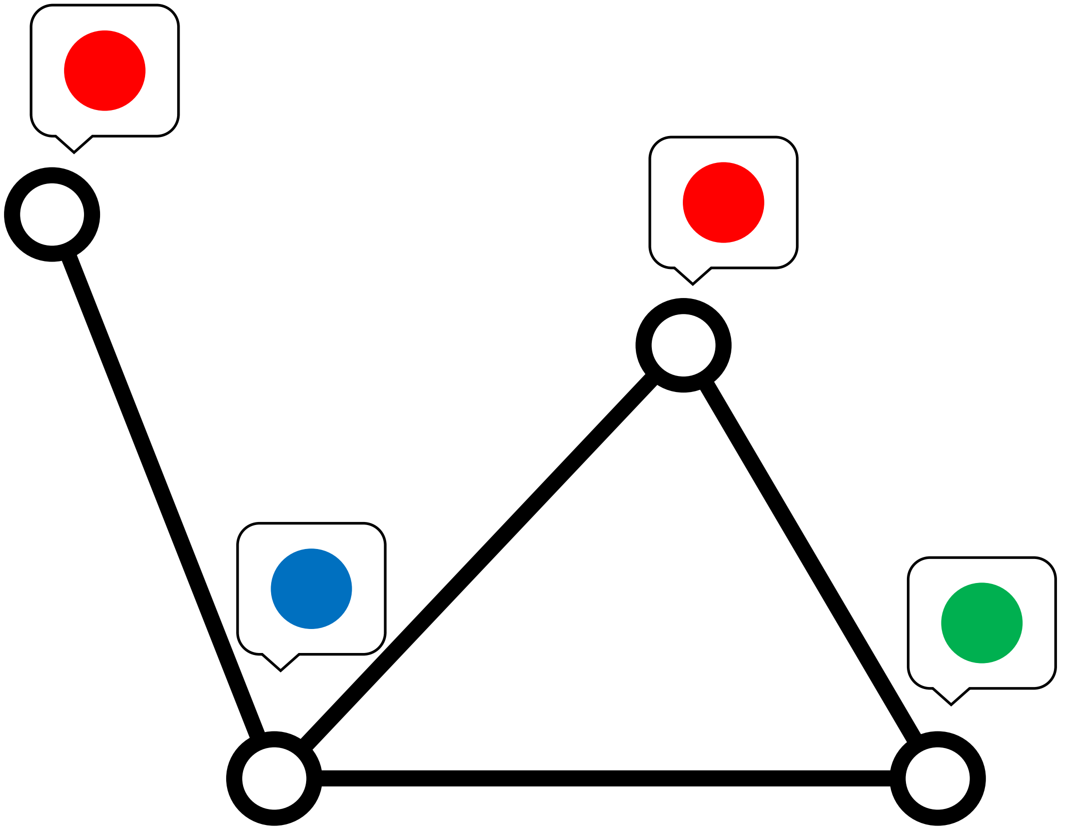


The Distributed LLL

- Input: dependency graph

- Additional input for each node E_i: the random variables that E_i depends on (and *how* E_i depends on them)

- Output of each node E_i: an assignment of the variables it depends on such that:
  1) it agrees with its neighbours
  2) the bad event E_i is avoided

Our result

- Moser and Tardos (2010): $O(\log^2 n)$

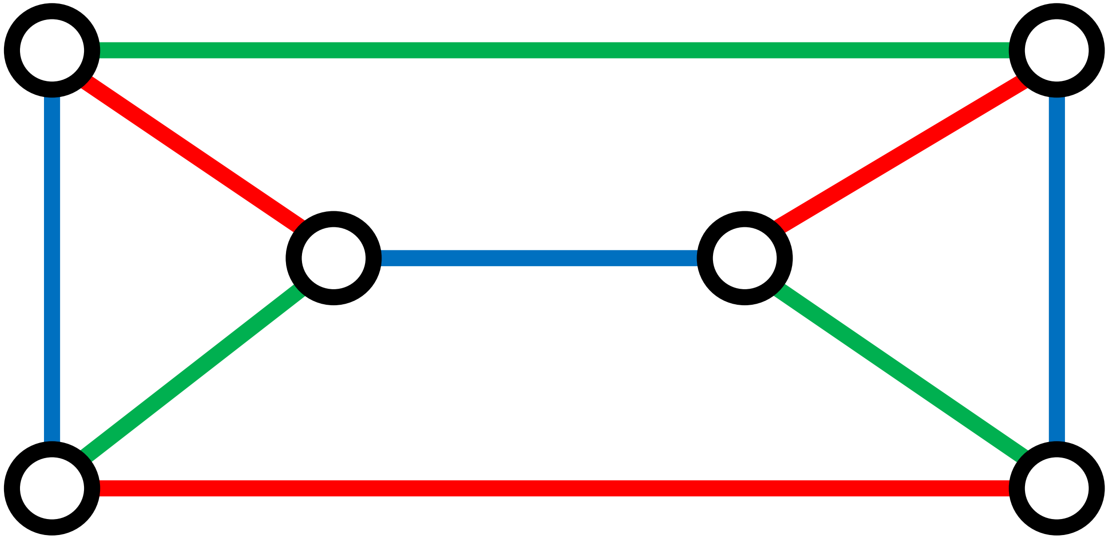
- Chung et al. (2014): $O(\log n)$ for bounded-degree graphs
  $\Omega(\log^* n)$

- $\Omega(\log\log n)$ (Monte-Carlo, w.h.p.)

Sinkless Orientation

- Input: edge d-coloured, d-regular graph

- Output of each node: non-conflicting orientations of the incident edges such that the node itself is not a sink



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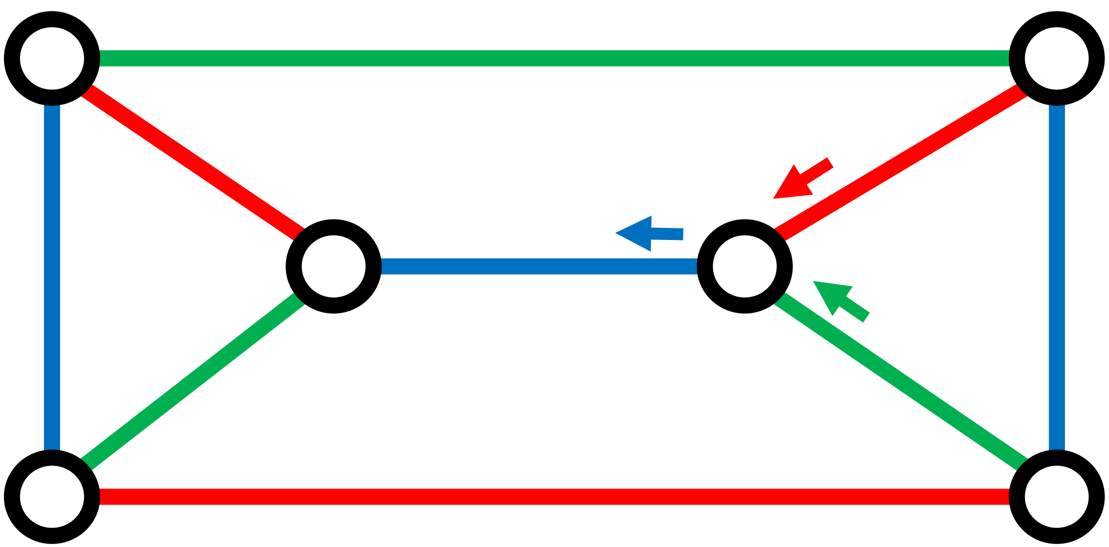

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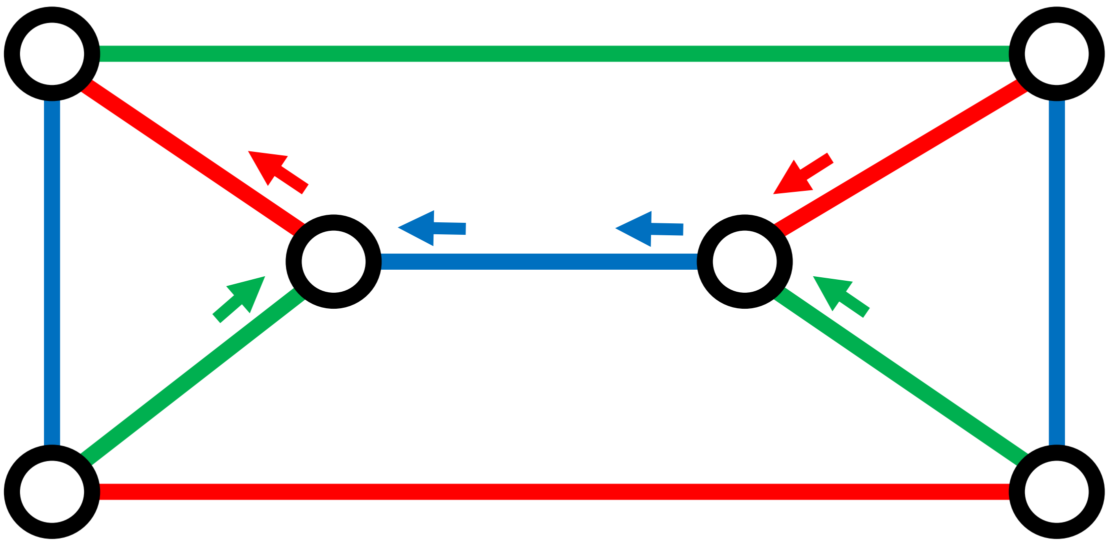

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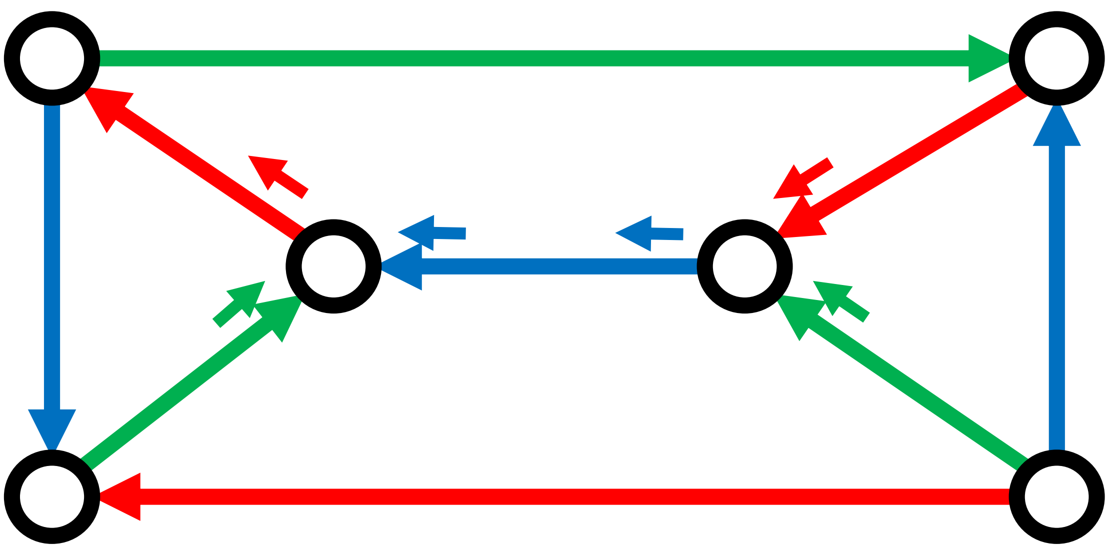
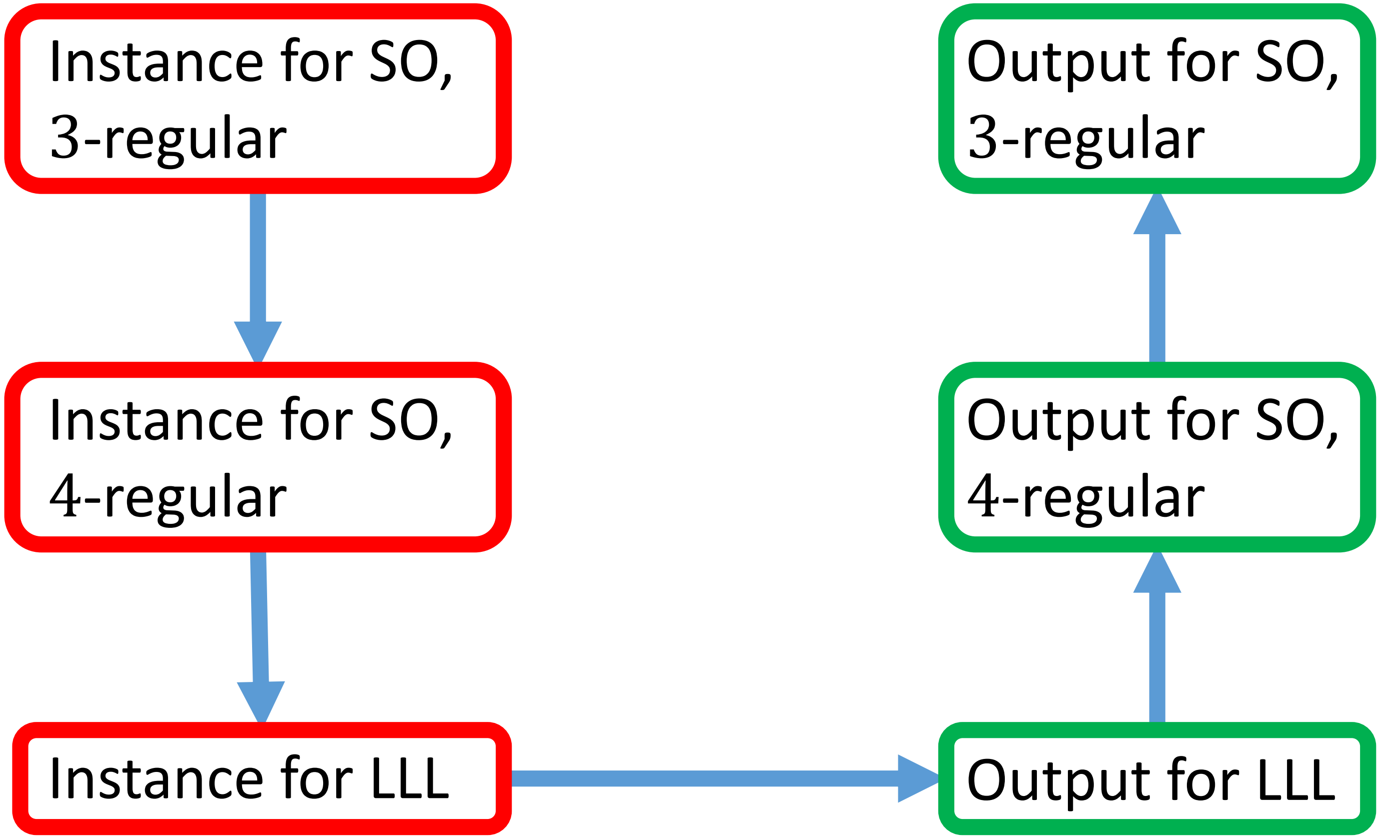


Reduction from SO to LLL



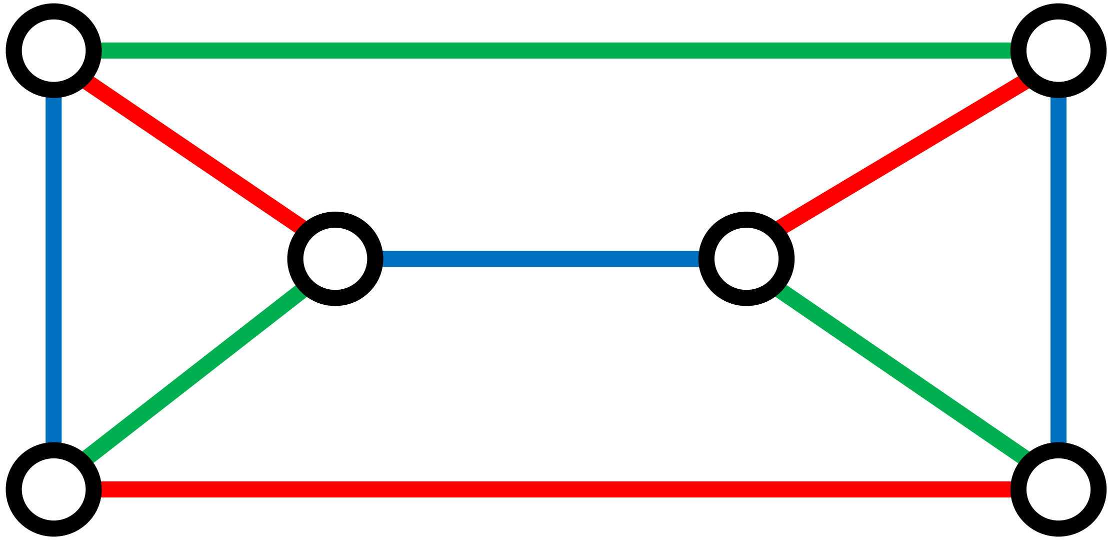
Sinkless Colouring

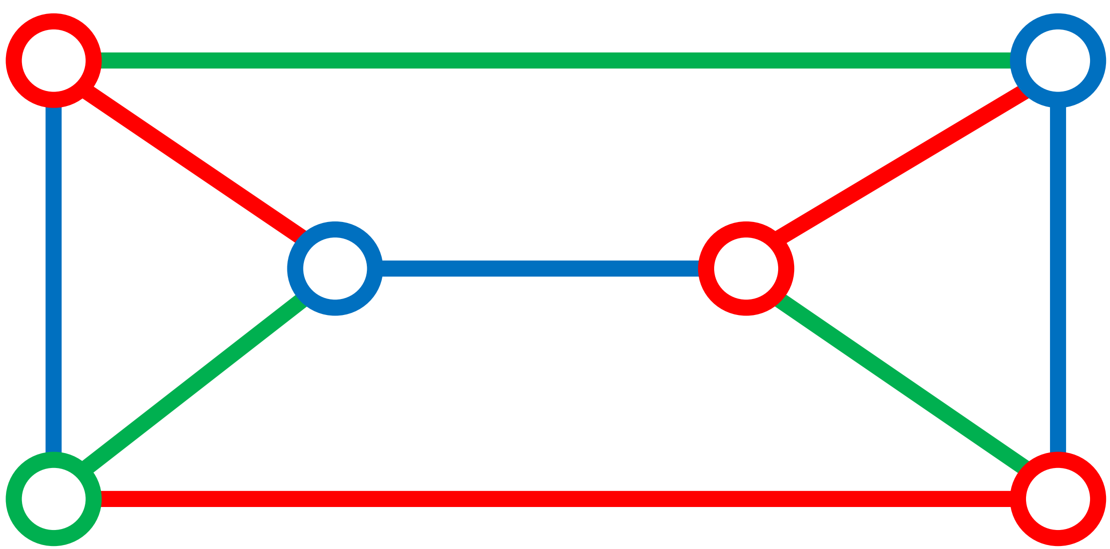
- Input: edge d-coloured, d-regular graph

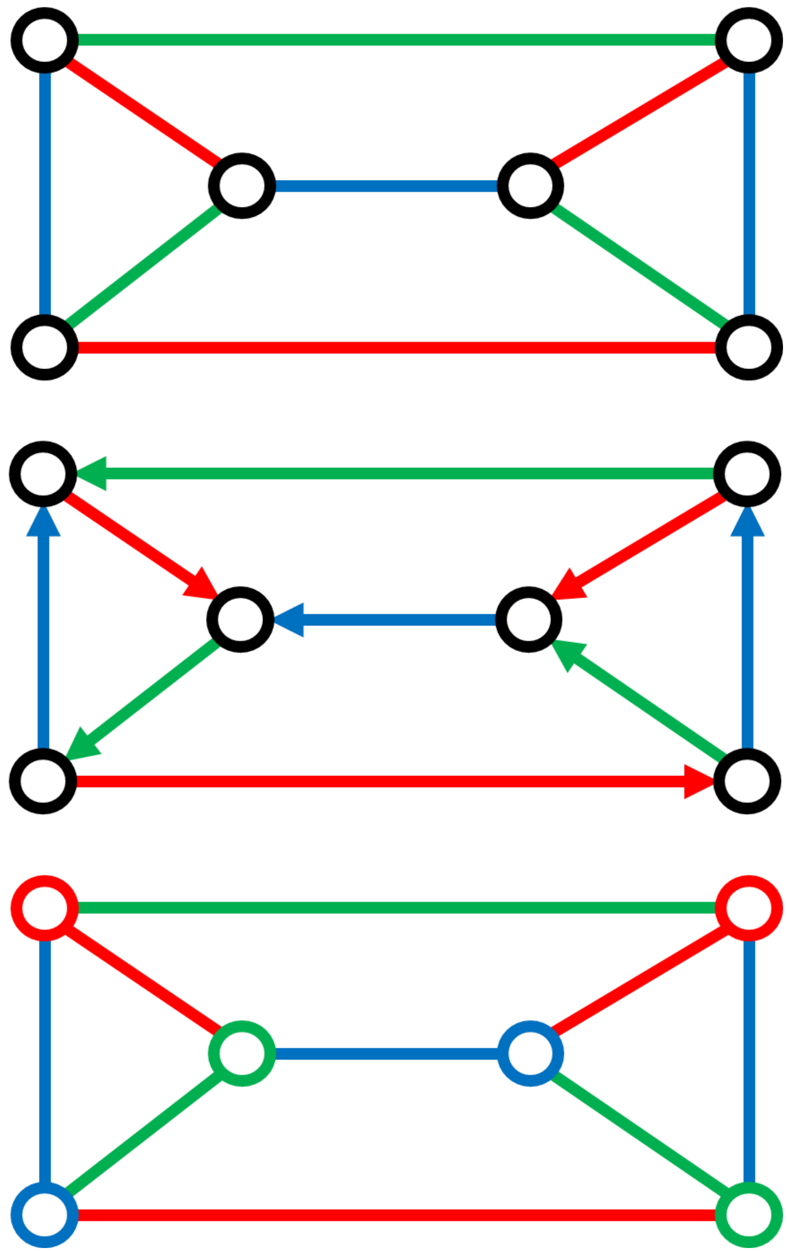
- Output of each node: one of the d colours such that no *forbidden configuration* occurs

Forbidden!

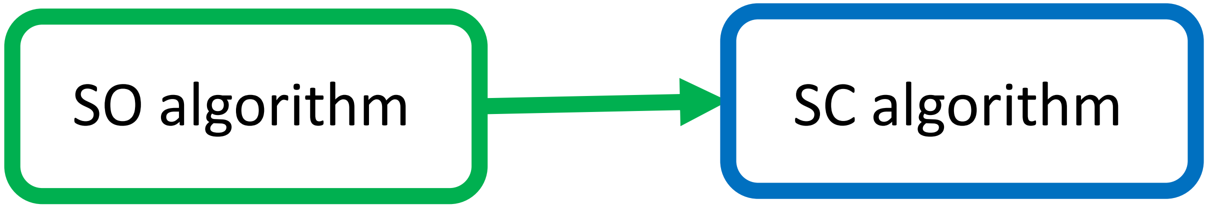
Fine!

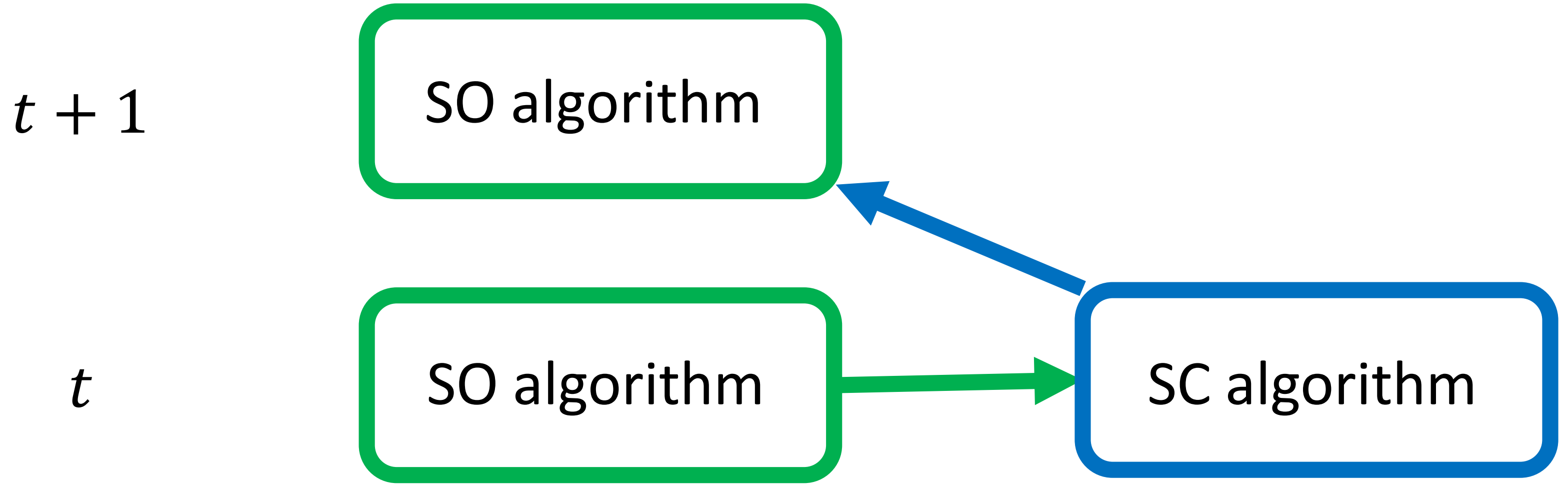


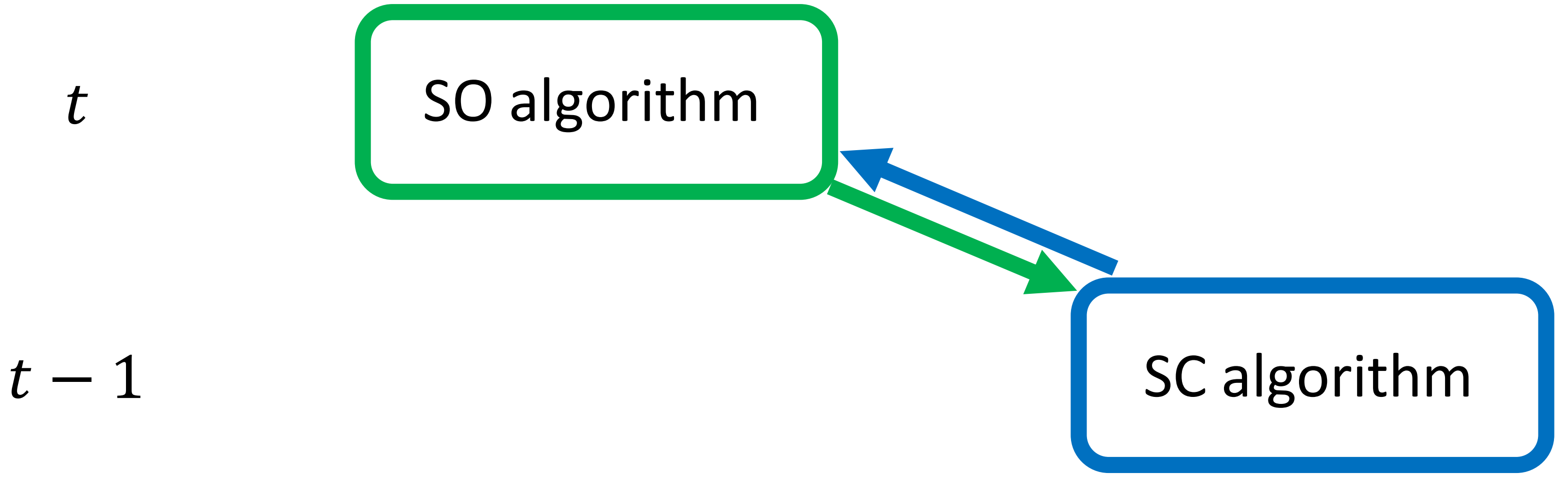




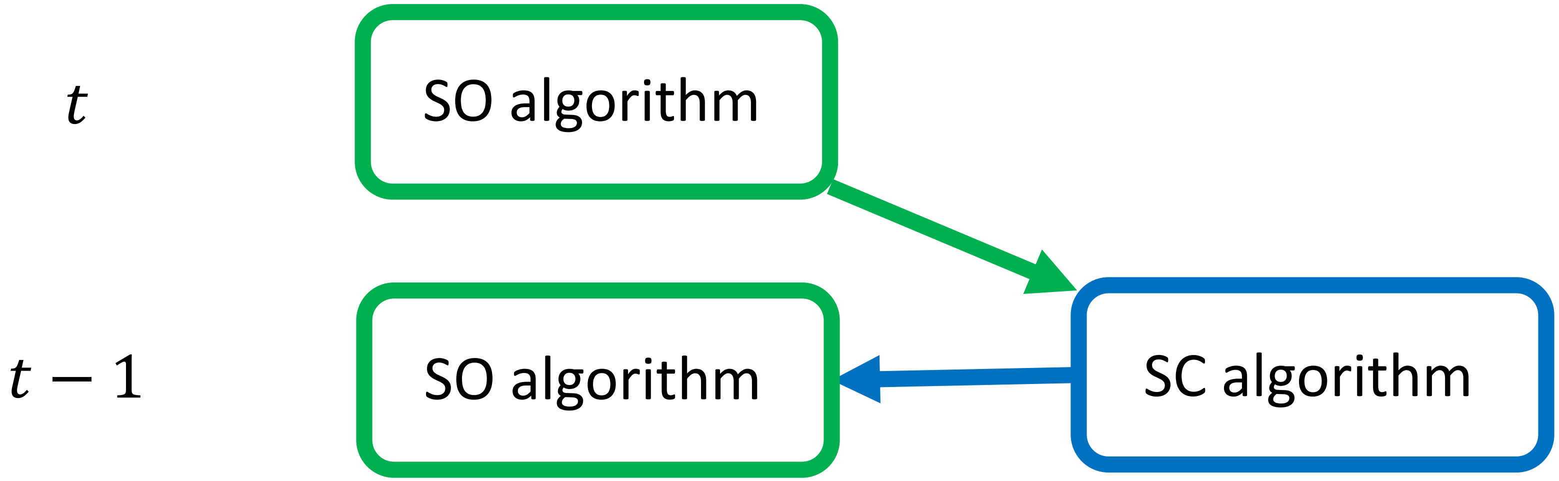
t

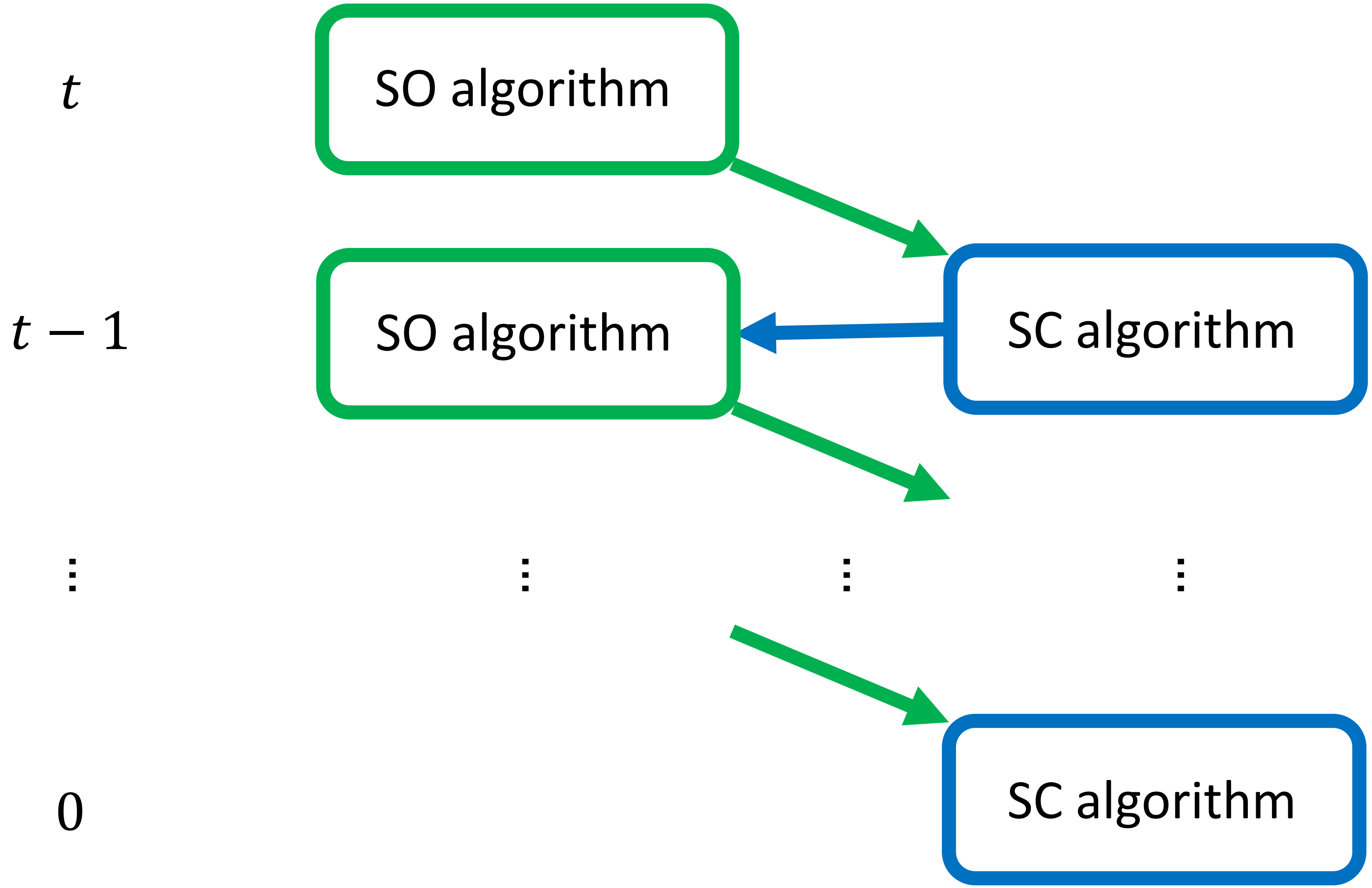



$t+1$
SO algorithm
t
SO algorithm
SC algorithm

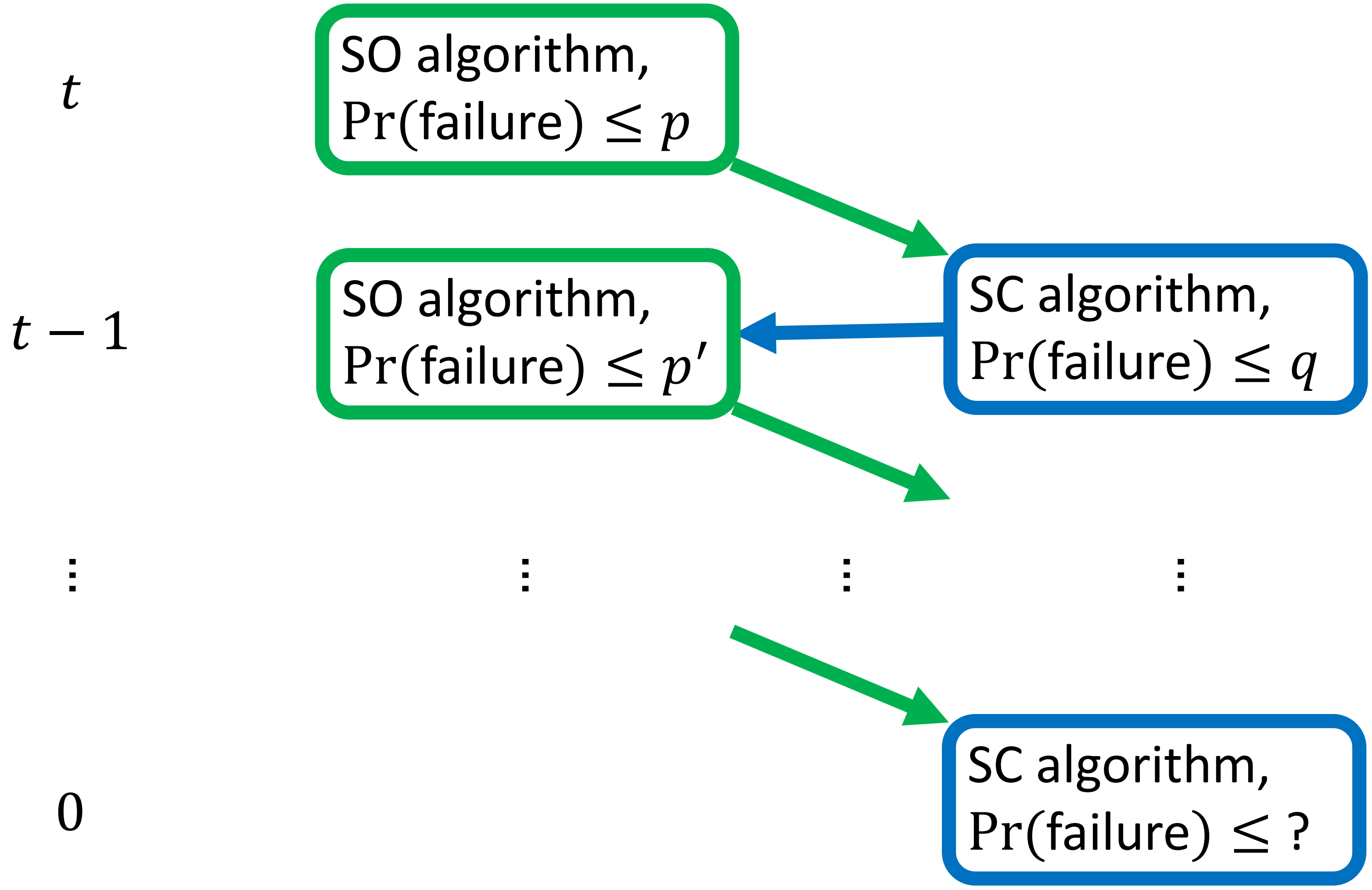


t
SO algorithm
$t-1$
SC algorithm

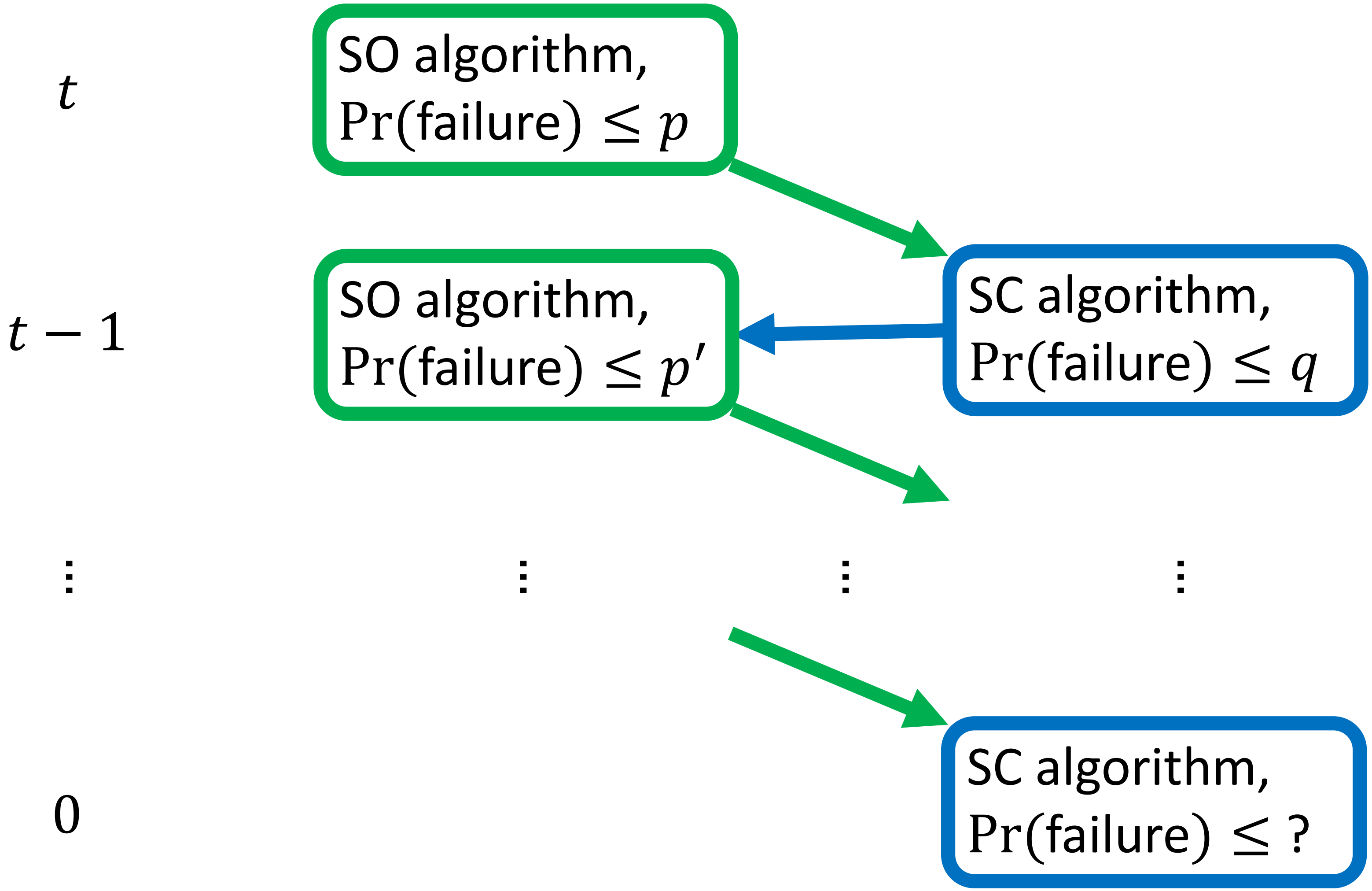

t
$t-1$
SO algorithm
SO algorithm
SC algorithm

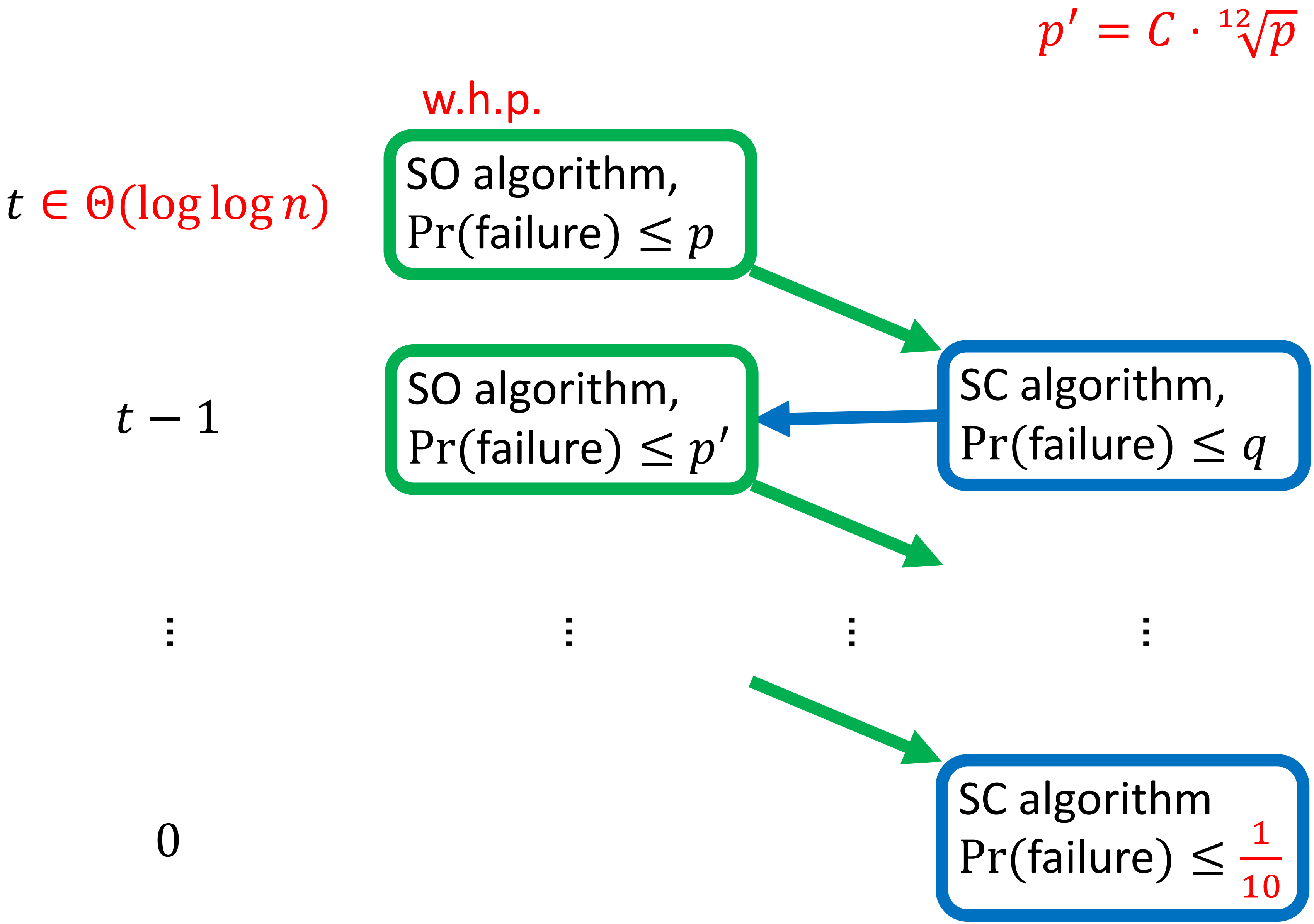


t
SO algorithm
$t-1$
SO algorithm
SC algorithm
$\vdots$
0
SC algorithm


t
SO algorithm,
Pr(failure) $\leq p$
$t-1$
SO algorithm,
Pr(failure) $\leq p'$
SC algorithm,
Pr(failure) $\leq q$
$\vdots$
0
SC algorithm,
Pr(failure) $\leq$?

$$p' = C \cdot \sqrt[12]{p}$$




$p' = C \cdot \sqrt[12]{p}$
w.h.p.
$t \in \Theta(\log \log n)$
SO algorithm,
Pr(failure) $\leq p$
$t - 1$
SO algorithm,
Pr(failure) $\leq p'$
SC algorithm,
Pr(failure) $\leq q$
$\vdots$
0
SC algorithm
Pr(failure) $\leq \frac{1}{10}$

Any Monte-Carlo algorithm for the distributed LLL that gives a correct output w.h.p. needs $\Omega(\log \log n)$ rounds.

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Any Monte-Carlo algorithm for finding a node d-colouring in d-regular, bipartite, $\Omega(\log n)$-girth graphs that gives a correct output w.h.p. needs $\Omega(\log \log n)$ rounds.

Chang et al. (2016)

The randomised time complexity of finding a node d-colouring in trees with maximum degree d is $\Theta(\log_{\mathrm{d}} \log n)$, the deterministic complexity is $\Theta(\log_{\mathrm{d}} n)$.

Backup Slides

The Constructive LLL

- Each E_i shares variables with at most d other events
- $\Pr(E_i) < p$ where $ep(d+1) \leq 1$

$\Rightarrow$ An assignment of the random variables that avoids all bad events can be found efficiently

Moser and Tardos, 2010

- Example: X_i binary

$$(X_1 \vee \neg X_2) \wedge (\neg X_1 \vee X_3 \vee \neg X_4) \wedge (X_2 \vee X_4) \wedge (\neg X_3 \vee X_4)$$

vbl$(E_1) = \{X_1, X_2\}$, vbl$(E_2) = \{X_1, X_3, X_4\}$,
vbl$(E_3) = \{X_2, X_4\}$, vbl$(E_4) = \{X_3, X_4\}$

$d = 3,\ p = \frac{1}{4}$

The Dependency Graph

- Nodes: events
- Edges: the events share a variable

- Example:

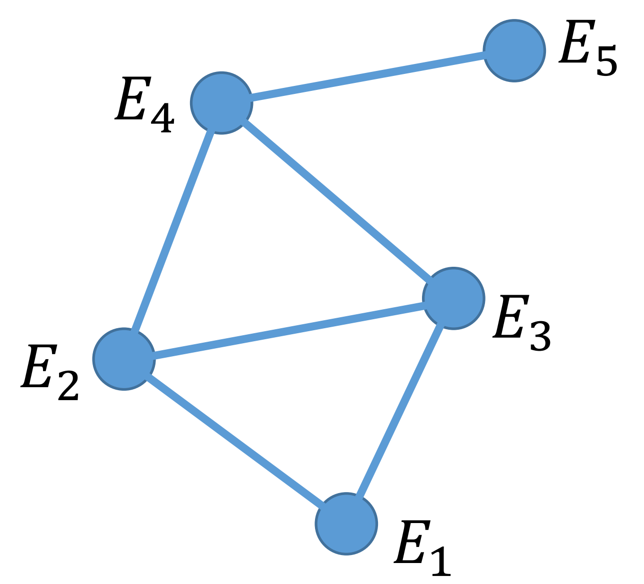
  $\mathrm{vbl}(E_1) = \{X_1\}$

  $\mathrm{vbl}(E_2) = \{X_1, X_2\}$

  $\mathrm{vbl}(E_3) = \{X_1, X_3\}$

  $\mathrm{vbl}(E_4) = \{X_2, X_3, X_4\}$

  $\mathrm{vbl}(E_5) = \{X_4\}$



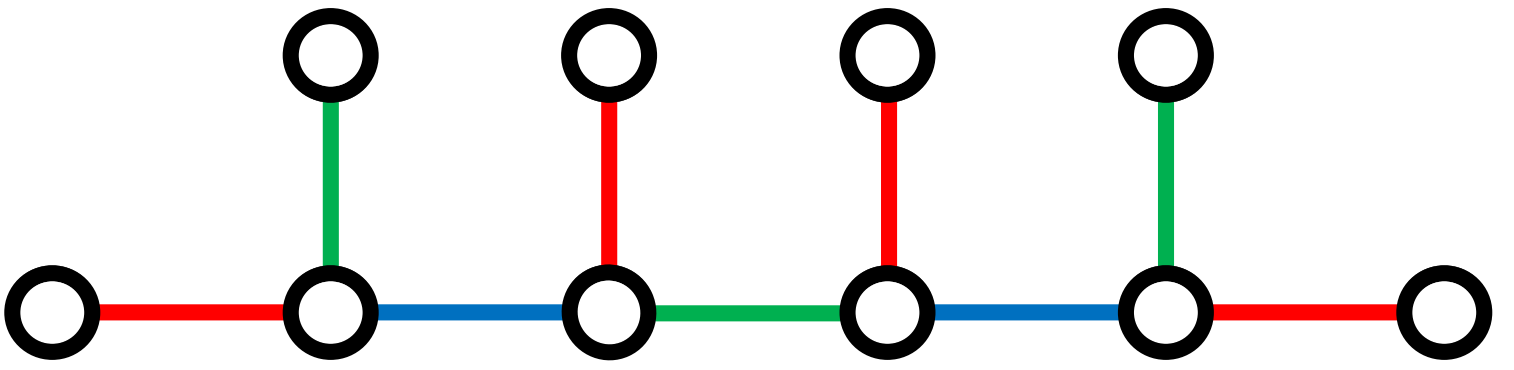
- Maximum degree d

Distributed Computing

- Input: simple undirected graph (+ some task-specific input)
- Nodes: computational entities
- Edges: communication channels

- Synchronous rounds
- In each round, each node ...
    1) sends an arbitrarily large message to each neighbour
    2) receives sent messages
    3) performs local computations
- Each node has to output a correct answer
- Time complexity: number of rounds (worst-case input)

Reduction from SO to LLL



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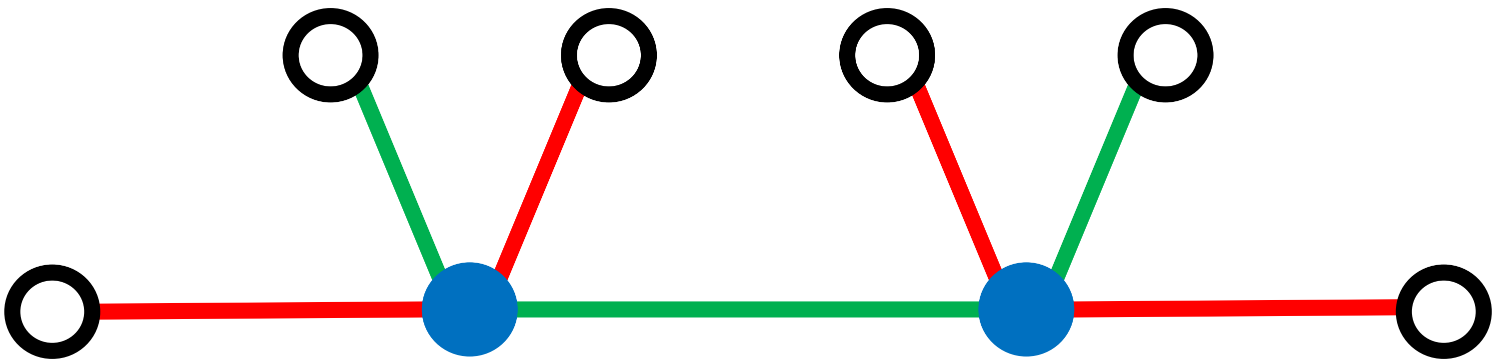

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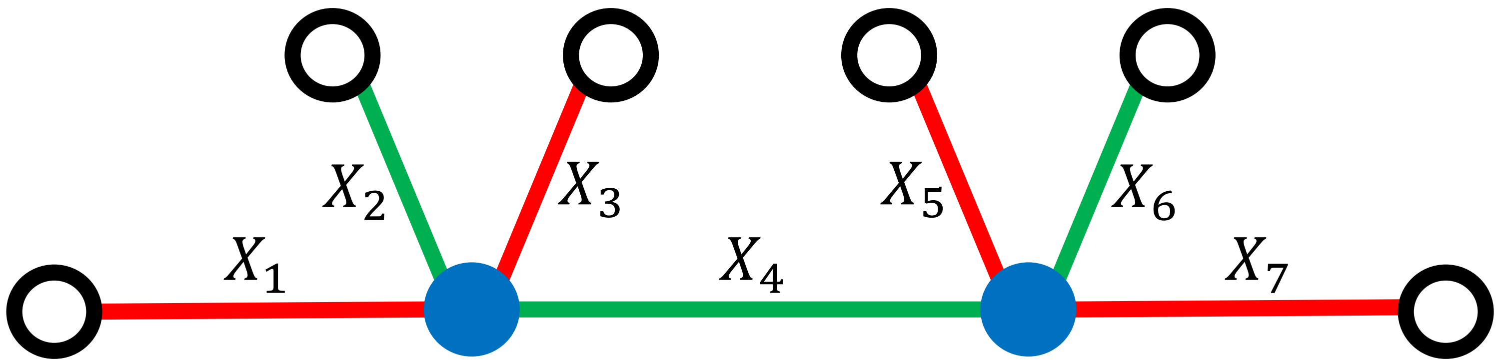

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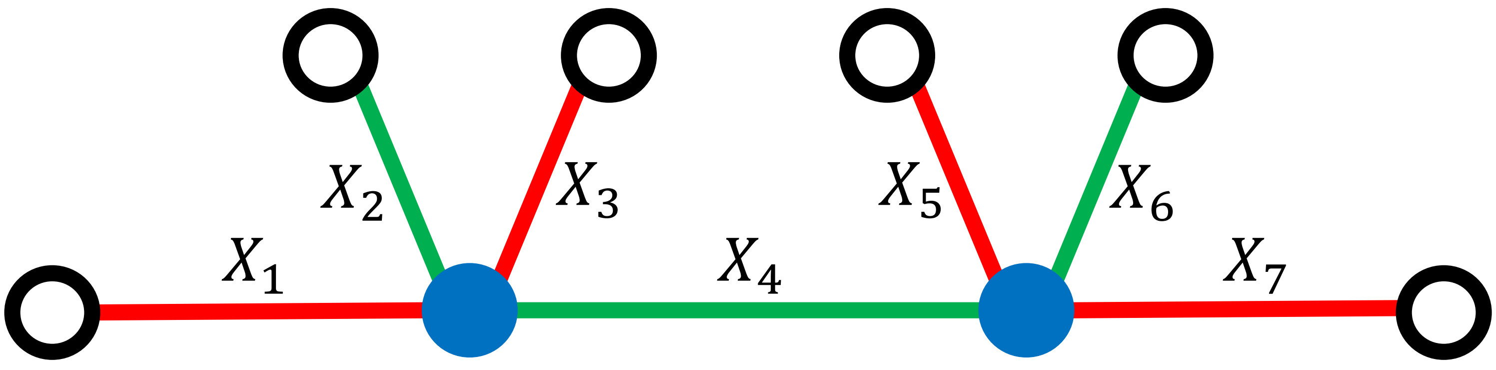
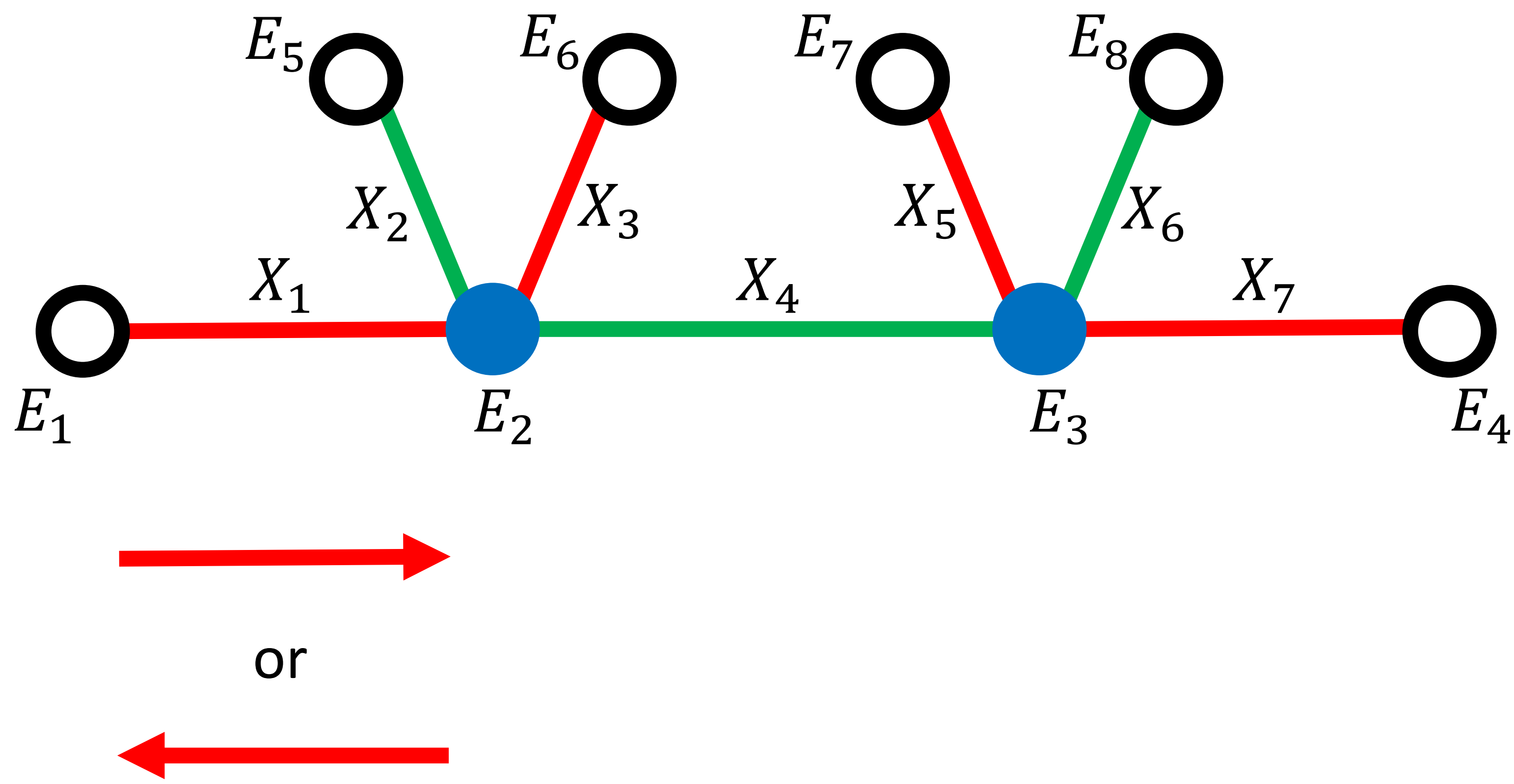


or

Reduction from SO to LLL



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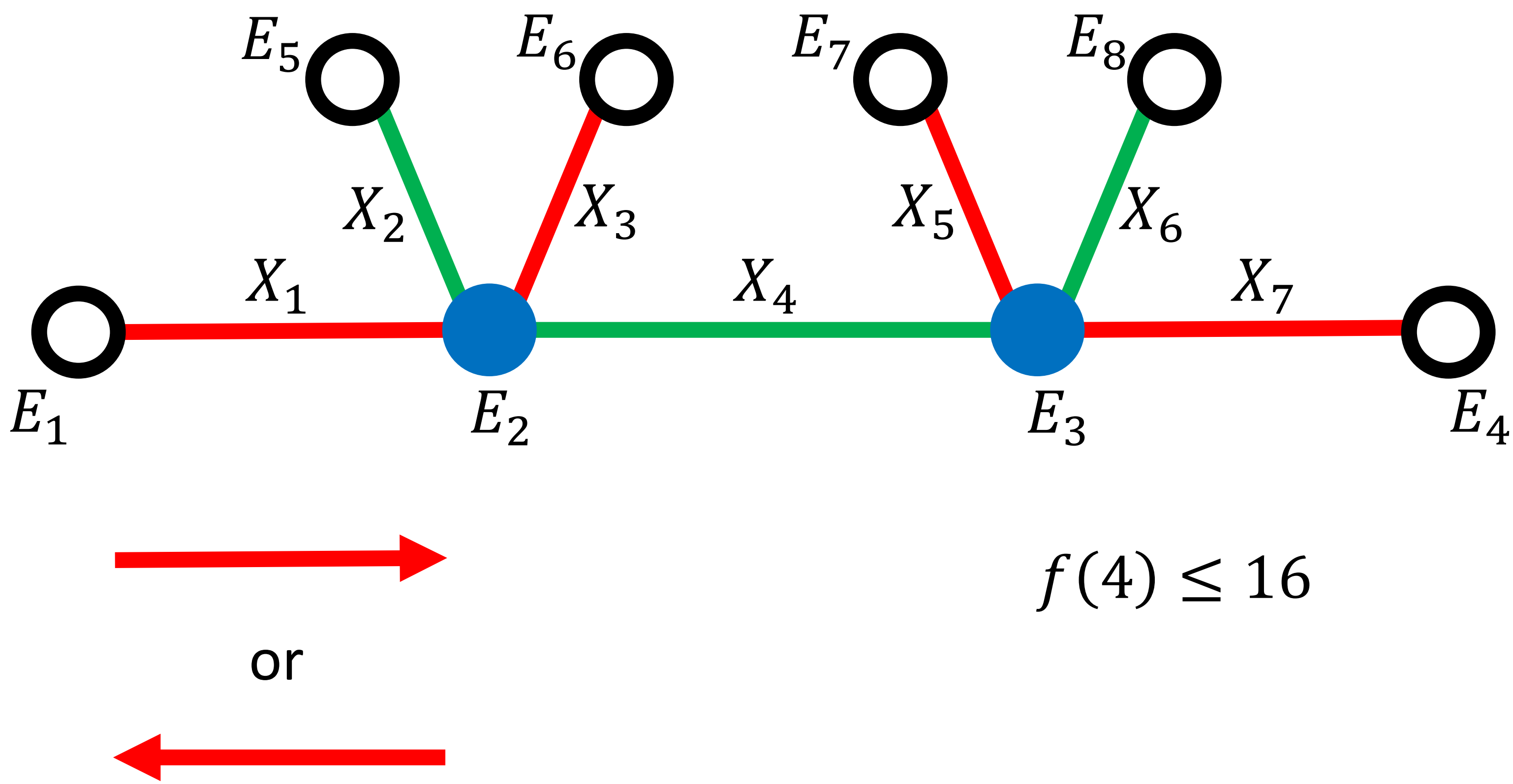
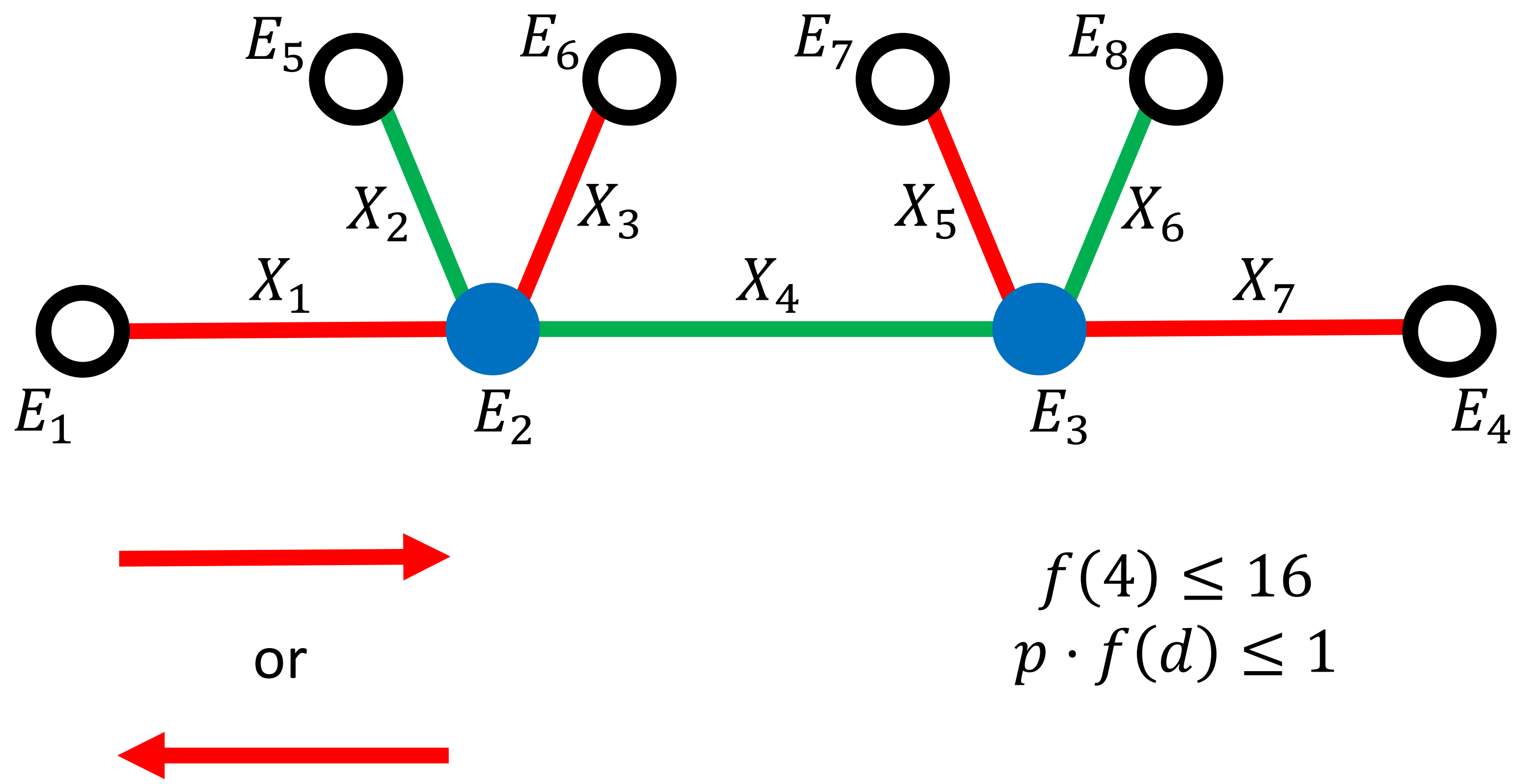


$f(4) \leq 16$

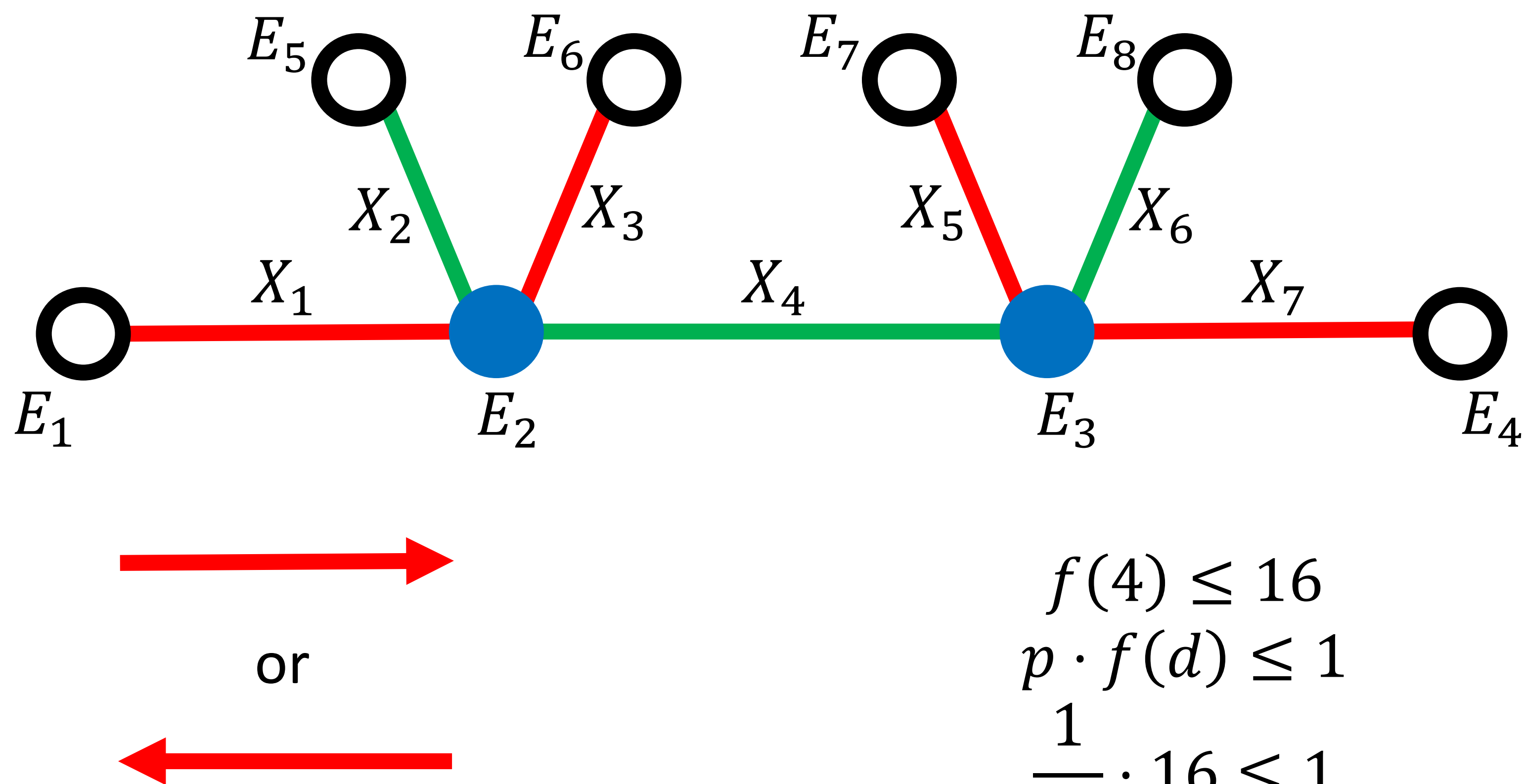
Reduction from SO to LLL



$f(4) \leq 16$

$p \cdot f(d) \leq 1$

Reduction from SO to LLL



$$f(4) \leq 16$$
$$p \cdot f(d) \leq 1$$
$$\frac{1}{16} \cdot 16 \leq 1$$

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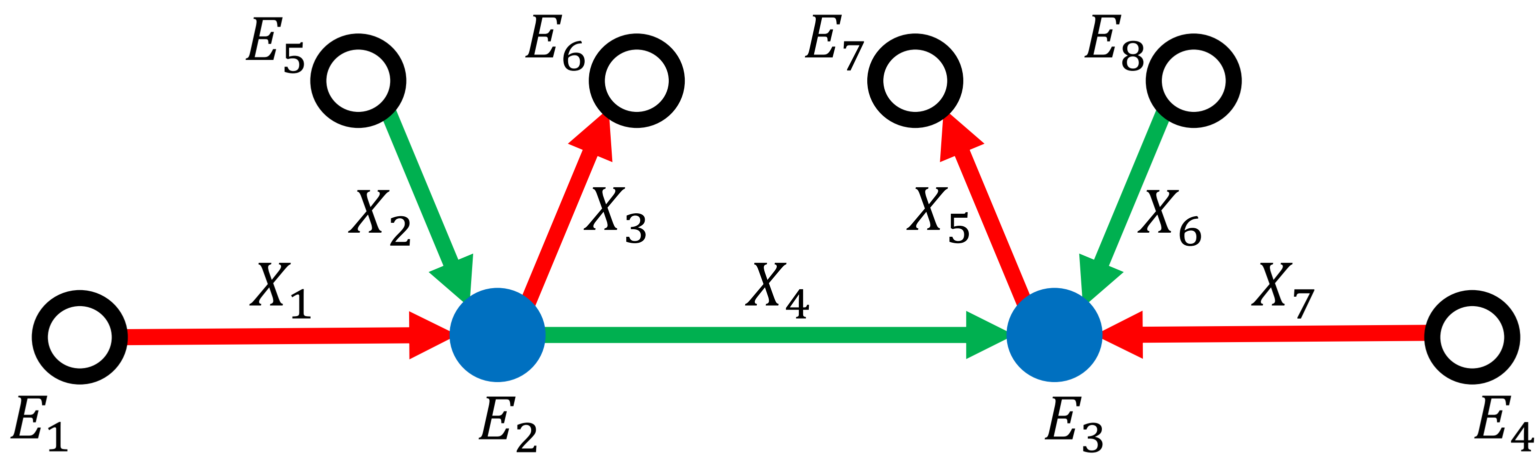

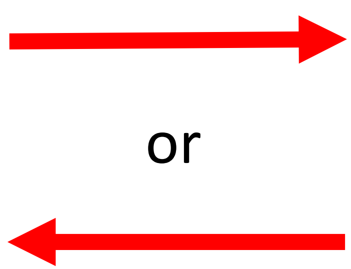

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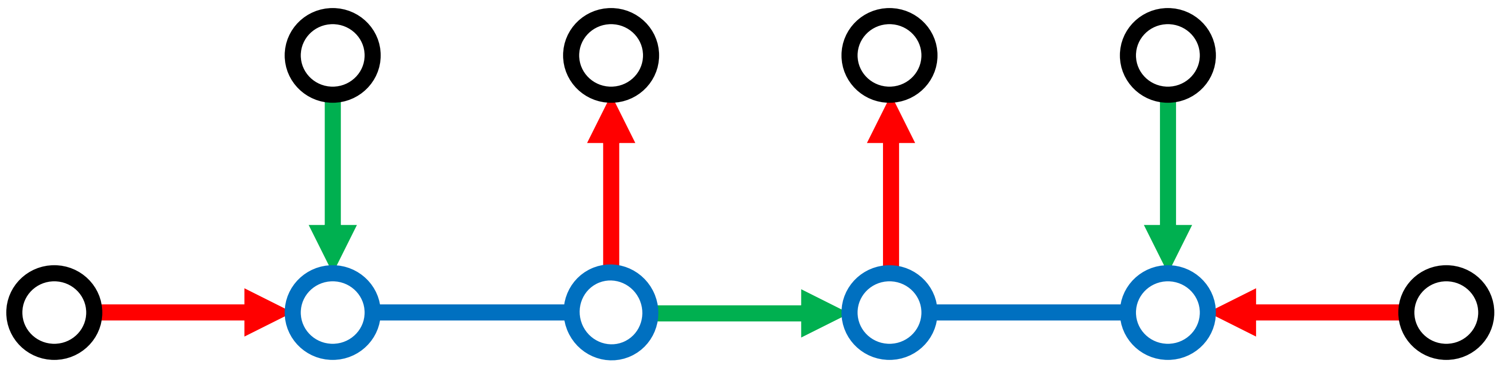

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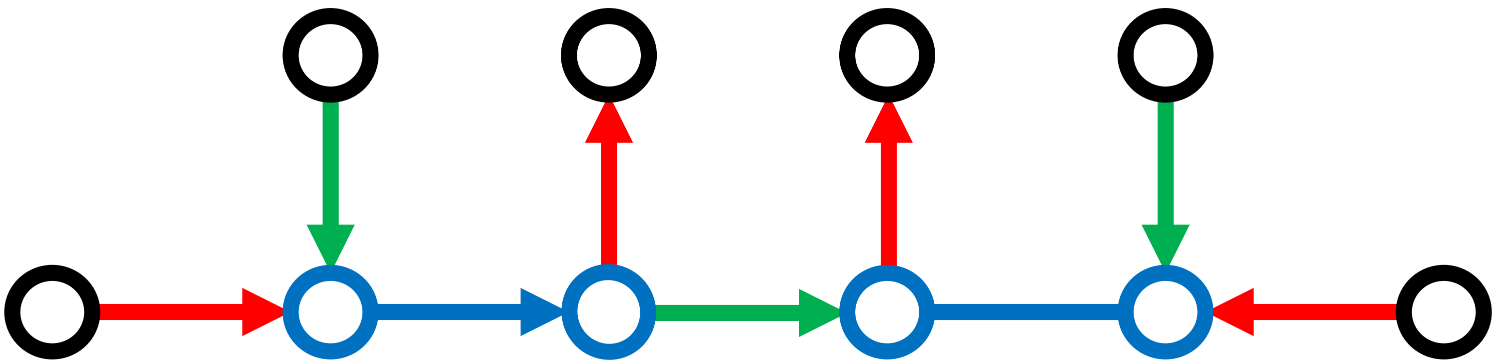

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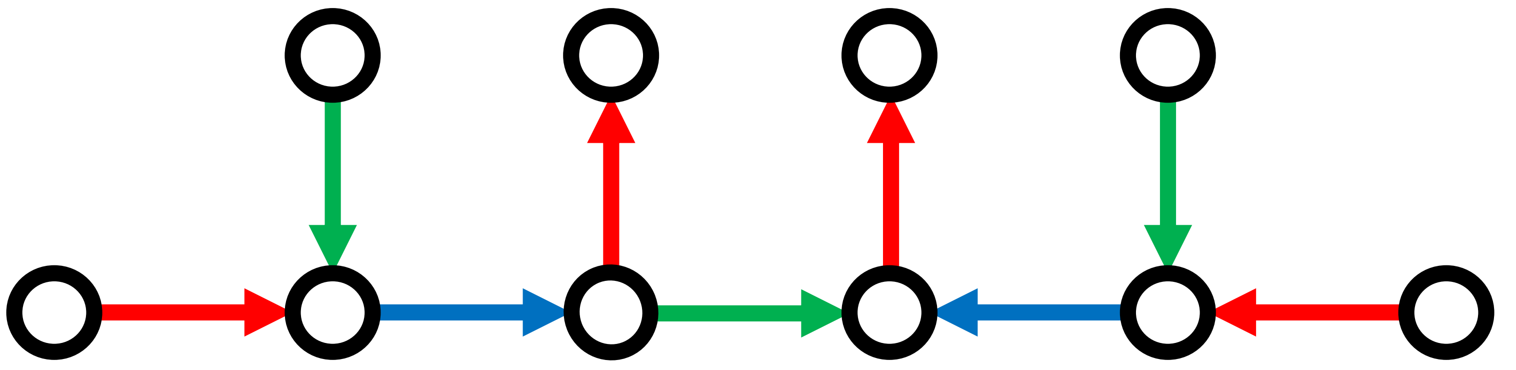

Technicalities

- Monte-Carlo algorithms, w.h.p.
- Girth $\geq 2t + 1$
- $d = 3$
- Failure probability $p_f(v)$ resp. $p_f(e)$