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Firefighting on the hexagonal grid

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ABSTRACT

The firefighter problem with k firefighters on an infinite graph G is an iterative graph process, defined as follows: Suppose a fire breaks out at a given vertex $v \in V(G)$ on Turn 1. On each subsequent even turn, k firefighters protect k vertices that are not on fire, and on each subsequent odd turn, any vertex that is on fire spreads the fire to all adjacent unprotected vertices. The firefighters' goal is to eventually stop the spread of the fire. If there exists a strategy for k firefighters to eventually stop the spread of the fire, then we say G is k-containable.

We consider the firefighter problem on the *hexagonal grid*, which is the graph whose vertices and edges are exactly the vertices and edges of a regular hexagonal tiling of the plane. It is not known if the hexagonal grid is 1-containable. In Gavenčiak et al. (2014), it was shown that if the firefighters have one firefighter per turn and one extra firefighter on two turns, the firefighters can contain the fire. We improve on this result by showing that even with only one extra firefighter on one turn, the firefighters can still contain the fire.

1. Introduction

The problem of firefighters on graphs studies the following iterative process: given a graph G, a subset of the vertices are initially on fire on turn 1. Then in alternating turns some vertices are protected by firefighters and the fire spreads to all unprotected vertices adjacent to a vertex on fire. Once a vertex has been protected by a firefighter it is protected for the remainder of the process. Similarly once a vertex is on fire it remains on fire.

More formally, let V be the set of vertices, let $F^{(t)}$ be the set of vertices on fire on turn t, and let $P^{(t)}$ be the set of protected vertices on turn t. Initially, on turn 1, $F^{(1)}$ is some non-empty subset of V, and $P^{(1)}$ is empty. On an even turn $2t$, we let $P^{(2t)}$ be the union of $P^{(2t-1)}$ and a subset of $V \setminus F^{(2t-1)}$ and let $F^{(2t)} = F^{(2t-1)}$. On an odd turn $2t+1$, where $t \geq 1$, we let $F^{(2t+1)}$ be $N[F^{(2t)}] \setminus P^{(2t)}$, the closed neighborhood of $F^{(2t)}$ except for the vertices in $P^{(2t)}$, and let $P^{(2t+1)} = P^{(2t)}$. For simplicity when we say a firefighter protects a vertex on turn t we assume this t to be even.

Let $t \in \mathbb{N}$. On turn t, we say that a vertex is **on fire** or **burning** if it is in $F^{(t)}$, **protected** if it is in $P^{(t)}$, and **unprotected** if it is in neither of these sets. We say a vertex is **saved** if it is impossible for it to ever be on fire. That is, a vertex is saved if it is in $P^{(t)}$ or in a component of the subgraph induced by $V \setminus P^{(t)}$ with no burning vertices. We say a vertex is **actively burning** if it is burning and has a neighbor that is unprotected.

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When firefighting on an infinite graph, we say the fire is **contained** if all but finitely many vertices are saved. We say an infinite graph given with a subset of vertices initially on fire is ***k*-containable** if the fire can be contained by protecting at most k vertices every even turn.

Problems related to k-containability also exist on finite graphs. For example, there is the NP-Complete decision problem of whether it is possible to save all vertices in a set S by protecting at most k vertices every even turn [4].

Firefighting on graphs can be used to model network spread, and can be used to understand the spread of computer viruses, misinformation, and infectious diseases. Indeed, similar problems arise in SIR epidemic models where a disease seeded at an initial set of vertices spreads through a network. Effects of vaccination programs in such models where vertices are granted immunity from infection have been extensively studied [1]. Conversely, the firefighters can also be thought of as an adversary; for example, the fires and the firefighters could model a broadcast signal and an adversary trying to censor it in a communication network. In this context, it would be good for the communication network to be robust against censorship. There is a significant body of existing related work, some of which is discussed in [2].

1.1. The hexagonal grid

The problem of k-containability has been studied on various infinite graphs. The infinite triangular grid, formed by tiling the plane with equilateral triangles and letting the corners be vertices, with a single initially burning vertex is conjectured to not be 2-containable [3]. Here, we focus on the infinite hexagonal grid, formed by tiling the plane with equilateral hexagons with the corners as vertices. It is conjectured that the hexagonal grid is not 1-containable, [6].

It is known that all orientations of the hexagonal grid are 1-containable as in a directed graph fire can only spread to out-neighbors, [5]. The following theorem suggests that if the hexagonal grid is not 1-containable, then it is "barely" not 1-containable.

Theorem 1.1 ([3]). *If it is possible to use an additional firefighter at two turns, $2t_1$ and $2t_2$ possibly with $2t_1 = 2t_2$, then one firefighter every turn is sufficient to contain the fire on the hexagonal grid with a single initially burning vertex.*

Our main contribution is an improvement to this result.

Theorem 1.2. *If it is possible to use an additional firefighter at a single turn, 2τ, then one firefighter every turn is sufficient to contain the fire on the hexagonal grid with a single initially burning vertex.*

Our theorem shows that the hexagonal grid conjecture, if true, is in some sense sharp; it would not be true if even a single extra firefighter was available. Our firefighting strategy, like the two-extra-firefighters strategy in [3], does not need to know in advance which turn the extra firefighters can be used.

2. One extra firefighter suffices

In this section, we prove Theorem 1.2. Our strategy is very similar to the one given in [3] to prove Theorem 1.1, however we optimize the strategy at certain points to contain the fire without a second extra firefighter.

For the strategy description and proof, we fix some notation. Let V be the set of vertices of the hexagonal grid. Let $\tau^* \in \mathbb{Z}_{>0}$ be such that at turn $2\tau^*$, two firefighters can be used. To simplify the proof, we wish to use our extra firefighter on turn 2τ, where

$$\tau = \begin{cases} \tau^* & \text{if } \tau \text{ is odd,} \\ \tau^* + 1 & \text{if } \tau^* \text{ is even.} \end{cases}$$

In this way, we can enforce that τ is odd, and if τ^* is even, we will play our extra firefighter in the place we would if we were given the firefighter on turn $2(\tau^* + 1) = 2\tau$ instead of turn $2\tau^*$.

Parts of our strategy are essentially the same as the strategy given in [3], but for completeness we provide all the details here. To be able to address vertices of the hexagonal grid, we draw the hexagonal grid on the Cartesian plane with regular hexagons, here the initial vertex on fire, f, is at the origin, and the grid is oriented such that there is a vertex adjacent to f directly above it, and that every edge of the graph has length 1. Note that throughout this section, every reference to distance will be distance in the hexagonal grid, not Euclidean distance. To be able to address vertices without using square roots or fractions, we make a change of coordinates: Let $i = \frac{2}{\sqrt{3}}x$, and let $j = 2y$. The vertex (i, j) corresponds to the point $(x, y) = (\frac{i\sqrt{3}}{2}, \frac{j}{2})$ on the Cartesian plane. See Fig. 1 for this mapping of the hexagonal grid on the Cartesian plane. Note that (i, j) is a vertex of the hexagonal grid if and only if $(i \bmod 2, j \bmod 6) \in \{(0, 0), (0, 2), (1, 3), (1, 5)\}$.

In the proof of our strategy, we often use the distance $\operatorname{dist}(f, v)$ from the initial fire to a given vertex v. Let $P_d = \{v \in V : \operatorname{dist}(f, v) = d\}$ be the set of vertices v at distance d from f. In Fig. 1, the sets P_d are marked by green and violet lines for $1 \le d \le 6$. Let $(d \bmod 2)$ refer to the remainder of dividing d by 2. Note that since we embed the hexagonal grid in the plane with f at the origin, we have that the distance $\operatorname{dist}(f, v) = d$ in the hexagonal grid if and only if

$$\max\left\{\frac{|2j - (d \bmod 2)|}{3}, |i| + \frac{|j + (d \bmod 2)|}{3}\right\} = d. \tag{1}$$

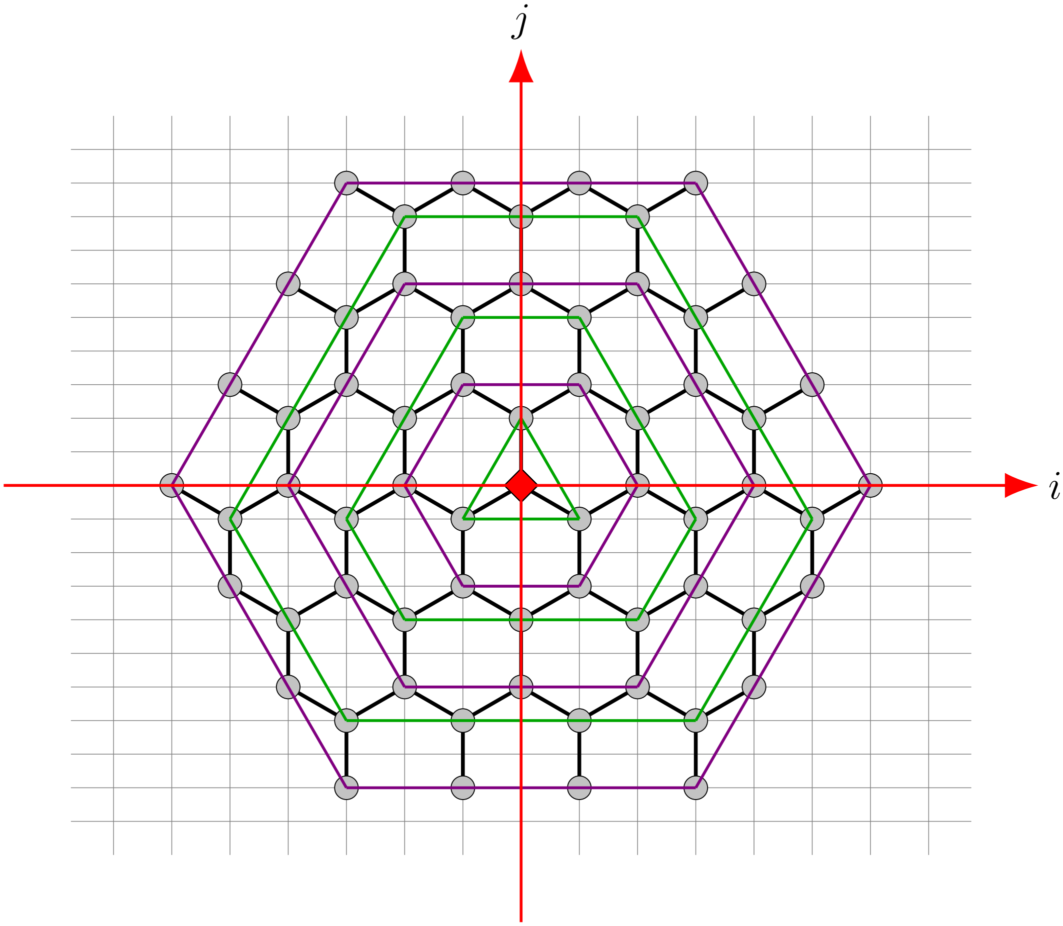


Fig. 1. The (i, j) coordinate system used throughout Section 2. $f = (0, 0)$ is the red vertex at the center. Its three neighbor vertices, starting from the one directly above, and in clockwise order, are $(0, 2), (1, -1)$, and $(-1, 1)$.

The proof that (1) characterizes points at distance d from the origin is straightforward but tedious, so we only give a sketch of the argument here. Let E_d be the points of V satisfying (1). It is straightforward to check that the E_d partition V, $E_0 = P_0$, and if $u \in E_{d_1}$ is adjacent to $v \in E_{d_2}$, then $|d_1 - d_2| = 1$. Using these facts and induction on d, we can show that $E_d = P_d$. One can see that $P_d \subseteq E_d$ by the fact that every vertex of P_d must have a neighbor in E_{d-1}, and thus must be in E_{d-2} or E_d, but are not in E_{d-2}. And to see that $E_d \subseteq P_d$, it suffices to show that every vertex of E_d has a neighbor in E_{d-1}. We can do this by checking cases that depend on the parity of d, the sign of i, and the sign of $j + (d \bmod 2)$. For instance, consider a vertex $(i, j) \in E_d$ with $j < 0$ and d even. This implies that $j \bmod 3 = 0$, so $(i, j+2)$ is a neighbor of (i, j). Finally, we see that $(i, j + 2) \in E_{d-1}$ since

$$\max\left\{\frac{|2(j+2) - ((d-1) \bmod 2)|}{3}, |i| + \frac{|(j+2) + ((d-1) \bmod 2)|}{3}\right\}$$
$$= \max\left\{\frac{|2j+3|}{3}, |i| + \frac{|j+3|}{3}\right\} = \max\left\{\frac{|2j|}{3} - 1, |i| + \frac{|j|}{3} - 1\right\} = d - 1.$$

The other cases can be checked similarly.

Our strategy can be broken down into the following steps (see Fig. 2 for a visual outline):

1. Before turn 2τ, build two protective rays, that if extended indefinitely would protect $\frac{2}{3}$ of the grid.
2. Advance the ray building by one extra step with the extra firefighter at turn 2τ.
3. Bend the protective rays to be parallel to each other. Grow a strip containing the fire with these parallel rays for a sufficiently long time.
4. Bend a ray into a spiral around a vertex c. The spiral will collide with the other ray.

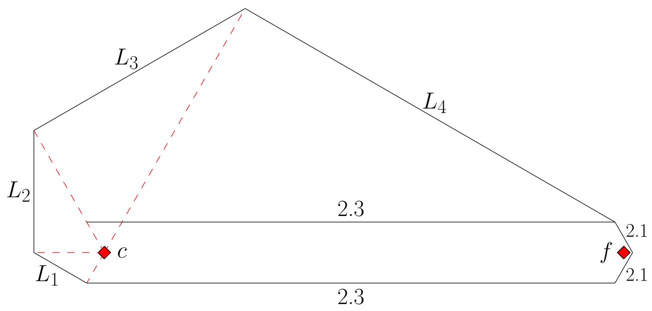


Fig. 2. Outline of the strategy. All angles are multiples of 30°. The portions of the protective rays built in Section 2.1, that if extended indefinitely would protect $\frac{2}{3}$ of the grid, are labeled 2.1. Once the rays have been bent to be parallel to each other, they are labeled after Section 2.3, where the parallel rays are built from the right to the left and the fire is constrained inside a strip. The bottom ray is eventually bent into a spiral around the vertex c, and L_1, L_2, L_3, L_4 are the segments of the spiral built in Section 2.4.

The improvement over the strategy described in [3] is given in the second and third parts. The first and last parts of our strategy are essentially the same as that in [3], but we describe them here for completeness.

2.1. Protecting two-thirds of the grid

For $0 \le k \le \frac{\tau-3}{2}$, on turn $4k+2$, protect $v_{2k+1} := (1-k, -1-3k)$; on turn $4k+4$, protect $v_{2k+2} := (1-k, 3+3k)$.

Observation 2.1. *Let $v_1, v_2, \ldots, v_{\tau-1}$ be the vertices protected in the manner described above. For all r with $1 \le r \le t-1$, we have* $\mathrm{dist}(f, v_r) = r$.

Proof. When $r = 2k+1$, we protect v_r, which is at $(1-k, -1-3k)$. Note that

$$\max\left\{\frac{|2(-1-3k)-1|}{3}, |1-k| + \frac{|(-1-3k)+1|}{3}\right\} = \max\{2k+1, |1-k|+k\} = r.$$

Furthermore, when $r = 2k+2$, we protect $(1-k, 3+3k)$, and

$$\max\left\{\frac{|2(3+3k)-0|}{3}, |1-k| + \frac{|(3+3k)+0|}{3}\right\} = \max\{2+2k, |1-k|+1+k\} = r,$$

so by Eq. (1), $\mathrm{dist}(f, v_r) = r$. □

Any burning vertex on turn $2j$ is at distance strictly less than j from f. And so, by Observation 2.1, we are permitted to protect the vertices described above. Fig. 3(a) illustrates this part of strategy.

2.2. Accelerating the ray building

At turn $2t$ when we receive the extra firefighter, we continue with the ray building strategy of the previous part, but accelerate it with the extra firefighter: We protect the vertices at $(1-\frac{\tau-1}{2}, -1-\frac{3(\tau-1)}{2})$ and $(1-\frac{\tau-1}{2}, 3+\frac{3(\tau-1)}{2})$ in turn $2t$, instead of just $(1-\frac{\tau-1}{2}, -1-\frac{3(\tau-1)}{2})$. Fig. 3(b) shows this step.

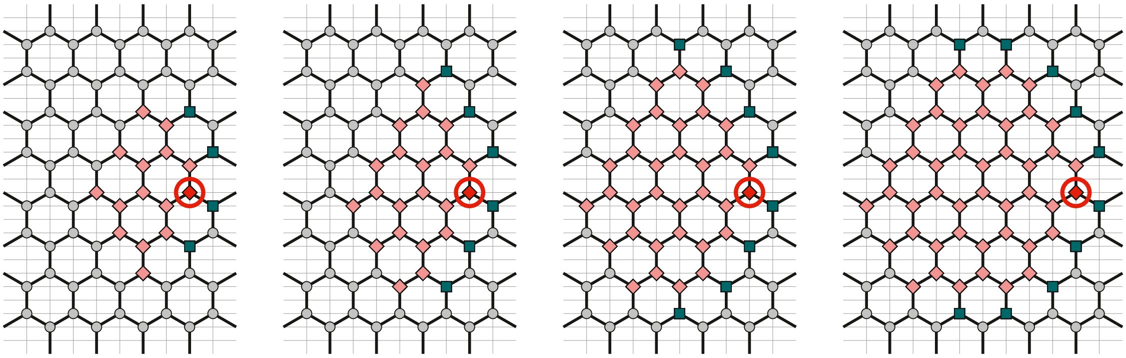
2.3. Restricting the fire to a strip

For $0 \le k \le \frac{15\tau+11}{2}$, on turn $2\tau+4k+2$ we protect $v_{\tau+2k+1} := (-\frac{\tau+1}{2} - 2k, -\frac{3(\tau+1)}{2})$, and on turn $2\tau+4k+4$ we protect $v_{\tau+2k+2} := (-\frac{\tau+1}{2} - 2k, 2+\frac{3(\tau+1)}{2})$.

Observation 2.2. *Let $v_{\tau+1}, v_{\tau+2}, \ldots, v_{16\tau+13}$ be the vertices protected in the manner described above. For all r with $\tau+1 \le r \le 16\tau+13$, we have* $\mathrm{dist}(f, v_r) = r$.

Proof. When $r = \tau+2k+1$, the vertex v_r is at $(-\frac{\tau+1}{2} - 2k, -\frac{3(\tau+1)}{2})$, so we have that

$$\max\left\{\frac{|2\,(-3(\tau+1)/2)-0|}{3}, \left|-\frac{\tau+1}{2} - 2k\right| + \frac{|(-3(\tau+1)/2)+0|}{3}\right\}$$
$$= \max\{\tau+1, \tau+2k+1\} = r,$$



(a) Turn $2\tau - 1$. (b) Turn $2\tau + 1$. (c) Turn $2\tau + 5$. (d) Turn $2\tau + 9$.

Fig. 3. Grid after turns $2\tau - 1$, $2\tau + 1$, $2\tau + 5$, and $2\tau + 9$; $\tau = 5$ and f is circled.

and when $r = \tau + 2k + 2$, v_r is at $(-\frac{\tau+1}{2} - 2k, 2 + \frac{3(\tau+1)}{2})$, and

$$\max\left\{\frac{|2\,(2+3(\tau+1)/2)-1|}{3}, \left|-\frac{\tau+1}{2}-2k\right| + \frac{|(2+3(\tau+1)/2)+1|}{3}\right\}$$
$$= \max\{\tau+2, \tau+2k+2\} = r.$$

Thus, by Eq. (1), $\mathrm{dist}(f, v_r) = r$. □

As before, these moves are permitted since on turn $2j$, we protect a vertex v with $\mathrm{dist}(f, v) = j$. Fig. 3(d) shows this effective bending of the rays. We claim that so far, we have constrained the fire to a "strip".

Lemma 2.3. *After the fire spreads at turn $32\tau + 27$, a vertex $v = (i, j)$ is on fire if and only if all of the following inequalities hold:*

- $\mathrm{dist}(f, v) \le 16\tau + 13$
- $-4 + 3i < j < 6 - 3i$
- $\frac{-3(\tau+1)}{2} < j < 2 + \frac{3(\tau+1)}{2}$

Proof. After turn $32\tau + 27$, the fire has spread exactly $16\tau + 13$ times, so the fire is completely contained inside $B(f, 16\tau + 13)$, where $B(f, r) := \{v \in V \mid d(f, v) \le r\}$ is the closed ball of radius r centered around the vertex f, which corresponds to the restriction imposed on the first bullet point above.

In Sections 2.1 and 2.2 , we protected every vertex of the form $(1 - k, -1 - 3k)$ and $(1 - k, 3 + 3k)$, for all k with $0 \le k \le \frac{(\tau-1)}{2}$, which corresponds exactly to vertices of the form (i, j) where $j = -4 + 3i$ and $j = 6 - 3i$ as i ranges from $1 - \frac{\tau-1}{2}$ to 1. Note that after turn 2τ, the fire has spread $\tau - 1$ times, so it was completely contained inside $B(f, \tau - 1)$, and the vertices we protected in Sections 2.1 and 2.2 separate $B(f, \tau - 1)$ into two regions, with f in the left region, so at turn 2τ, the points on fire all satisfy the second bullet point above.

In Section 2.3, we protected the vertices $(-\frac{\tau+1}{2} - 2k, -\frac{3(\tau+1)}{2})$ and $(-\frac{\tau+1}{2} - 2k, 2 + \frac{3(\tau+1)}{2})$ for all k with $0 \le k \le \frac{15\tau+11}{2}$, which correspond to vertices (i, j) that satisfy the first and second bulletpoints above, and that have $j = \frac{-3(\tau+1)}{2}$ or $j = 2 + \frac{3(\tau+1)}{2}$. The set of vertices protected after turn $32\tau + 26$ again separate $B(f, 16\tau + 13)$ into two regions, and the region containing f is characterized by the second and third bulletpoints above, so these points and no other points are on fire after the fire spreads on turn $32\tau + 27$. □

Fig. 3 provides an example of the early part of our strategy when $\tau = 5$.

2.4. Building a protective spiral

We will now bend the lower ray we built in Section 2.3 into a clockwise spiral around the vertex $c = (-15\tau - 13, 0)$. Our goal is to construct this spiral in a way that it eventually collides with the upper ray, thus containing the fire. We first note where the actively burning vertices are.

Observation 2.4. *After the fire spreads at turn $32\tau + 27$, a vertex (i, j) is actively burning if and only if (i, j) is at distance τ from c and $\frac{-3(\tau+1)}{2} < j < 2 + \frac{3(\tau+1)}{2}$ while $i < -15\tau - 13$.*

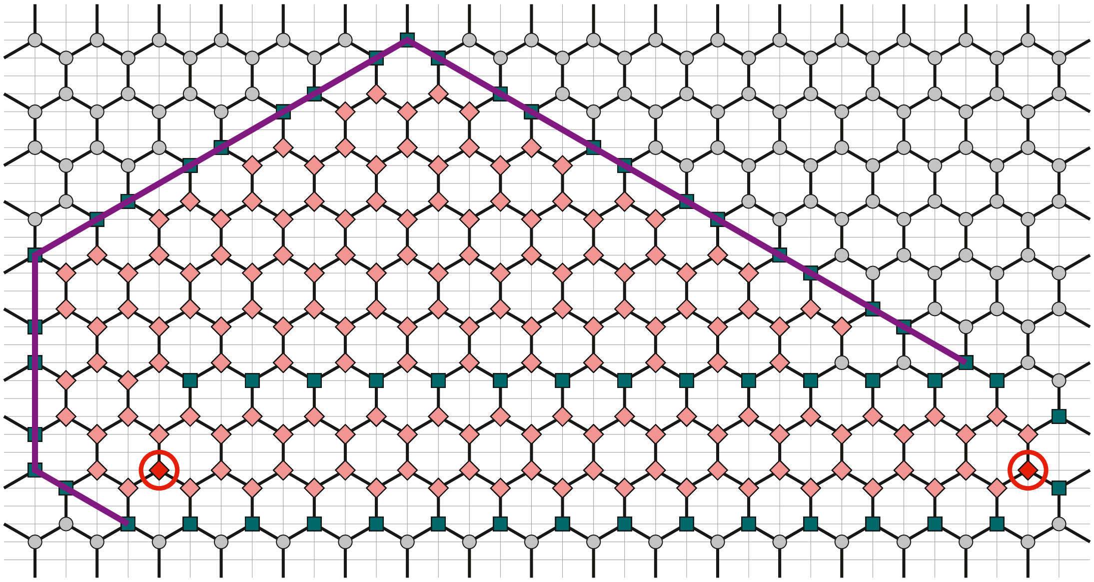


Fig. 4. A complete picture of our strategy for $\tau = 1$, once the last vertex in L_4 has been protected. f and c are circled. The line segments L_1, L_2, L_3, and L_4 are drawn in violet. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Proof. By Lemma 2.3, the vertices that are actively burning are the vertices on the line segments $(-16\tau - 13, -1)$ to $(\frac{-31\tau-25}{2}, \frac{3\tau+1}{2})$, and $(-16\tau - 13, -1)$ to $(\frac{-31\tau-27}{2}, \frac{-3\tau+1}{2})$. All of these vertices are at distance τ from c. $\square$

When we build the protective spiral, we will do so in such that way that for every $s \geq 0$, on turn $2s + 32\tau + 28$ we protect a vertex $v_{s+16\tau+14}$ with $\text{dist}(v_{s+16\tau+14}, c) = \tau + s + 1$. By Observation 2.4, on turn $32\tau + 28$ every actively burning vertex is at distance τ from c, so the placement of $v_{s+16\tau+14}$ will be a legal move. We start by noting that a shifted variant of Eq. (1) holds: Given a vertex $v = (i, j)$, the distance $\text{dist}(c, v) = d$ in the hexagonal grid if and only if

$$\max\left\{\frac{|2j - (d \bmod 2)|}{3}, |i + 15\tau + 13| + \frac{|j + (d \bmod 2)|}{3}\right\} = d. \tag{2}$$

The spiral is built by initially bending the lower ray 30° clockwise, and then bending it 60° clockwise at three later points in time. The initial 30° bend will occur at the vertex $(\frac{-31\tau-27}{2}, \frac{-3\tau-3}{2})$, while the subsequent 60° bends will occur at the vertices $(-17\tau - 15, 0)$, $(-17\tau - 15, 6\tau + 6)$, and $(-11\tau - 9, 12\tau + 12)$.

We first protect the vertices on the line segment L_1, which runs from $(\frac{-31\tau-27}{2}, \frac{-3\tau-3}{2})$ to $(-17\tau - 15, 0)$, then we protect the vertices on the line segment L_2, from $(-17\tau - 15, 0)$ to $(-17\tau - 15, 6\tau + 6)$, then the vertices on the line segment L_3, from $(-17\tau - 15, 6\tau + 6)$ to $(-11\tau - 9, 12\tau + 12)$, and finally, the vertices on the line segment L_4, from $(-11\tau - 9, 12\tau + 12)$ to $(\frac{-\tau-3}{2}, \frac{3\tau+9}{2})$. Note that if we were to extend this final line segment by one vertex, this vertex would be $(\frac{-\tau-1}{2}, \frac{3\tau+7}{2}) = (1 - \frac{\tau-1}{2}, 3 + \frac{3(\tau+1)}{2})$, which was protected in Section 2.2. Hence, as long as the vertices along these line segments are indeed legal moves, the spiral has collided with the upper ray we built in Sections 2.2 and 2.3 , so we have successfully contained the fire. We now list the specific vertices which will be protected at each step so we can verify that they indeed are at the correct distances from c. Fig. 4 shows the end state we will reach after protecting the last vertex in L_4.

Note that L_1 consists of $\tau + 2$ vertices, so for $0 \leq k \leq \frac{\tau-1}{2}$, on turn $4k + 32\tau + 28$, we will protect $v_{2k+16\tau+14} := (\frac{-31\tau-27}{2} - 3k, \frac{-3\tau-3}{2} + 3k)$ and on turn $4k + 32\tau + 30$, we will protect $v_{2k+16\tau+15} := (\frac{-31\tau-31}{2} - 3k, \frac{-3\tau+1}{2} + 3k)$. Then finally on turn $34\tau + 30$, we protect $v_{17\tau+15} := (-17\tau - 15, 0)$.

Then the line segment $L_2 \setminus L_1$ consists of $2\tau + 2$ vertices, so for each $0 \leq k \leq \tau$, on turn $4k + 34\tau + 32$, we protect $v_{2k+17\tau+16} := (-17\tau - 15, 6k+2)$, and on turn $4k + 34\tau + 34$, we protect $v_{2k+17\tau+17} := (-17\tau - 15, 6k+6)$. This culminates when we protect $v_{19\tau+17} = (-17\tau - 15, 6\tau + 6)$ on turn $38\tau + 34$.

Now, the line segment $L_3 \setminus L_2$ has $4\tau + 4$ vertices, so for $0 \leq k \leq 2\tau + 1$, on turn $4k + 38\tau + 36$, we protect $v_{2k+19\tau+18} := (-17\tau - 13 + 3k, 6\tau + 8 + 3k)$ and on turn $4k + 38\tau + 38$, we protect $v_{2k+19\tau+19} := (-17\tau - 12 + 3k, 6\tau + 9 + 3k)$. We protect the last vertex on L_3 on turn $46\tau + 42$ where we protect $v_{23\tau+21} = (-11\tau - 9, 12\tau + 12)$.

Finally, the line segment $L_4 \setminus L_3$ has $7\tau + 5$ vertices, so for $0 \leq k \leq \frac{7\tau+3}{2}$, on turn $4k + 46\tau + 44$, we protect $v_{2k+23\tau+22} := (-11\tau - 8 + 3k, 12\tau + 11 - 3k)$, and on turn $4k + 46\tau + 46$, we protect $v_{2k+23\tau+23} := (-11\tau - 6 + 3k, 12\tau + 9 - 3k)$. This indeed finishes with vertex $v_{30\tau+26} = (\frac{-\tau-3}{2}, \frac{3\tau+9}{2})$. Note that if we extended L_4 by one more vertex, we would arrive at

$(\frac{-\tau-1}{2}, \frac{3\tau+7}{2}) = v_{\tau+2}$, so L_4 intersects the upper ray, containing the fire. See Fig. 2 for a depiction of the line segments L_1, L_2, L_3 and L_4.

Now that we have described the remaining vertices we will protect, we will show that they indeed have the correct distance from c, implying that all these moves were legal moves.

Lemma 2.5. *The distance* $\mathrm{dist}(c, v_{s+16\tau+14}) = \tau + s + 1$ *for every* $0 \le s \le 14\tau + 12$.

Proof. It suffices to verify Eq. (2) for each point with the correct value of d. We will do so for the vertices on L_1 to show how this could be done, but omit the remaining calculations for brevity.

The vertices in L_1 correspond to $0 \le s \le \tau + 1$. When $s = 2\ell$ for some ℓ, the vertex $v_{s+16\tau+14} = (\frac{-31\tau-27}{2} - 3\ell, \frac{-3\tau-3}{2} + 3\ell)$, so

$$\max\left\{\frac{|2((-3\tau-3)/2+3\ell)-0|}{3}, \left|\left(\frac{-31\tau-27}{2}-3\ell\right)+15\tau+13\right| + \frac{|((-3\tau-3)/2+3\ell)+0|}{3}\right\}$$
$$= \max\{\tau+1-2\ell, \tau+1+2\ell\} = \tau+s+1,$$

and when $s = 2\ell + 1$, the vertex $v_{s+16\tau+14} = (\frac{-31\tau-31}{2} - 3\ell, \frac{-3\tau+1}{2} + 3\ell)$, so

$$\max\left\{\frac{|2((-3\tau+1)/2+3\ell)-1|}{3}, \left|\left(\frac{-31\tau-31}{2}-3\ell\right)+15\tau+13\right| + \frac{|((-3\tau+1)/2+3\ell)+1|}{3}\right\}$$
$$= \max\{\tau-2\ell, \tau+2\ell+2\} = \tau+s+1.$$

All other vertices in $L_2 \cup L_3 \cup L_4$ can be similarly verified. □

Lemma 2.5 shows that each move in Section 2.4 was legal, which completes the proof of Theorem 1.2.

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