Lower Bounds for the Capture Time: Linear, Quadratic, and Beyond

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Abstract. In the classical game of Cops and Robbers on graphs, the capture time is defined by the least number of moves needed to catch all robbers with the smallest amount of cops that suffice. While the case of one cop and one robber is well understood, it is an open question how long it takes for multiple cops to catch multiple robbers. We show that capturing $\ell \in \mathcal{O}(n)$ robbers can take $\Omega(\ell \cdot n)$ time, inducing a capture time of up to $\Omega(n^2)$. For the case of one cop, our results are asymptotically optimal. Furthermore, we consider the case of a superlinear amount of robbers, where we show a capture time of $\Omega\left(n^2 \cdot \log(\ell/n)\right)$.

1 Introduction

This paper brings you back to your childhood, when you played the game of tag with your friends. Particularly interesting is a team version of tag sometimes known as jail, chase, manhunt, smee, or, as in this paper, cops and robbers. In cops and robbers, children are split into two teams, the cops and the robbers, where cops need to touch robbers, in order to jail them. If all children run at approximately the same speed, and the playground is suitably obstructed, the game becomes exciting, and cops usually need to cooperate in order to block possible escape paths of the robbers. Are there playgrounds (graphs) where the cops need a long time to catch all the robbers? This is the central open question we will investigate in this work.

The analytical study of these games on graphs is still relatively young. Breisch [9] first discussed searching for a lost person in a cave in 1967, followed by a formalization by Parsons [22,23] a decade later. The work of Quilliot [25] and Nowakowski and Winkler [21] introduced a game of pursuit-evasion on graphs, today commonly known as *Cops and Robbers*: A cop has to catch a robber, with both alternating in moves along edges. Aigner and Fromme [1] allowed multiple players into the game and showed that in any planar graph, three cops suffice to win. These articles spawned a rich field of interest, with plenty of further work, we refer to the book of Bonato and Nowakowski [7] for an in-depth overview and to [2,8,15] for recent surveys.

There are two central questions in the game of Cops and Robbers: First, how long will these cops need to catch the robbers, i.e., what is the *capture time*?

Secondly, how many cops are needed to catch the robbers, i.e., what is the *cop number*?

The case of one cop and robber is well understood, with the graphs where one cop suffices being characterized [21,25], and the time needed to capture one robber being at most $n-4$ in $n \geq 7$ vertex graphs [14]. With multiple cops and robbers, much is still unknown. For the first question, the best current result states that if k cops suffice, then they can capture a single robber in at most n^{k+1} time [3]. Already for one cop this bound is off by a factor of n. For the second question, $\mathcal{O}\left(n/\left(2^{(1-o(1))\sqrt{\log n}}\right)\right)$ cops always suffice [13,18,26] and there are graphs where $\Omega(\sqrt{n})$ cops are needed [24], but it is unclear what the exact bound is. Meyniel conjectured in 1985 that $\mathcal{O}(\sqrt{n})$ cops always suffice [12].

It is an open question what a good capture time lower bound would be for more than a single cop and robber, cf., e.g., [7]. It seems that so far, essentially only the cases of *i*) the cartesian product of two trees and *ii*) the d-dimensional hypercube have been successfully investigated for just one robber. For *i*), the capture time is half the diameter of the graph [19], while for *ii*), it is $\Theta(d \ln d)$, i.e., polylogarithmic in $n = 2^d$ [6].

After discussing further related work in the following Subsection 1.1 and a formal model in Section 2, we start with the case of one cop and any $\ell \in \mathcal{O}(n)$ robbers in Section 3, and prove that the capture time is $\Theta(\ell \cdot n)$.

In Section 4, we investigate the reversed case of k cops and one robber. As it turns out, the k cops might need $\Omega(n)$ time to capture a single robber, just like in the case of one cop.

Afterwards, we study the case of many cops and many robbers in Section 5, where we show that for k cops and any $\ell \in \mathcal{O}(\sqrt{n/k})$ robbers, the cops need at least $\Omega(\ell \cdot n)$ time to capture all robbers in general graphs. Furthermore, we discuss a superlinear number of robbers and show that the time to capture them all can be as high as $\Omega\left(n^2 \cdot \log\left(\ell/n\right)\right)$.

1.1 Further Related Work

The capture time density of a graph can be defined as as the ratio of the capture time to the number of vertices. Bonato et al. extended this notion to infinite graphs, or more precisely to limits of chains of induced subgraphs, and showed that the density can take any value from 0 to 1 for a single cop and robber [5]. It can be tested in polynomial time if fixed $k \in \mathbb{N}$ is the cop number of a graph [3], with these graphs being characterized in [10]. Nonetheless, determining the cop number of a graph is EXPTIME-complete [16].

Many more variants of Cops and Robbers are considered in the literature (e.g., can the cops win if they do not start too far away from the robber [4], applications to compact routing [17], or how to contain worm attacks in networks [11]), we again refer to [2,7,8,15] for an even further overview.

2 Model

The game of Cops and Robbers is a pursuit-evasion game played on undirected graphs $G = (V, E)$ with $|V| = n$ and diameter of D. We denote the set of $k \in \mathbb{N}$ cops as $C = \{p_1, p_2, \ldots, p_k\}$ and the set of $\ell \in \mathbb{N}$ robbers as $R = \{r_1, r_2, \ldots, r_\ell\}$. Throughout this paper, all graphs are assumed to be connected, finite, and, as standard in the literature, reflexive (i.e., each vertex has one self-loop, which is the same as allowing the cops and robbers to stand still).

The game proceeds in rounds, where each round consists of first the cops making a move and then the robbers making a move. In round 0, the cops make a move by placing each cop on a vertex and then the robbers make a move by placing each robber on a vertex. In round 0 and every further round, vertices can be shared by an arbitrary number of cops and robbers. For all subsequent rounds $i \geq 1$, first, a cop move consists of moving each cop along an incident edge, with second, a robber move defined by moving each robber along an incident edge. Both the cops and robbers have perfect information, i.e., they know the whole graph and every previous move played.[1]

A robber is caught if a cop shares its occupied vertex. Once robber r_i is captured, he is removed from the game, i.e., the cops need not to guard the robbers that are already captured. Should at least k cops be needed to catch all robbers on G, then the graph is called *k-copwin* with a *cop number* $c(G) = k$.

For $c(G) = k$ and $\ell = 1$, we define the *capture time* $capt(G, k, 1)$ as the smallest number of moves needed for the k cops to catch the robber, no matter what strategy the robber employs. Note that by this formulation, the capture time $capt(G, k, 1)$ is only defined on graphs where k cops can actually catch a robber. In a similar fashion, $capt(G, k, \ell)$ is the smallest number of moves needed for the k cops to catch all ℓ robbers. Let r_i be the i-th robber to be caught and define $capt(G, k, r_i)$ as the smallest number of moves needed for the k cops to catch the first i robbers.

3 One Cop, Many Robbers

We start with the case of one cop, many robbers. Gavenčiak showed in 2010 that $n - 4$ is the maximum capture time for one cop and one robber if $n \geq 7$ [14].[2] Observe that after catching one robber, the cop could go back to her starting position in at most diameter D moves and then catch the next robber in at most $n - 4$ moves. This gives us an upper bound for the capture time:

Observation 1 *Let G be a 1-copwin graph. Then, $capt(G, 1, \ell) \in \mathcal{O}(\ell \cdot n)$.*

Our next step will be to show a matching lower bound:

Theorem 2. *Let $n \geq 12$. Then, for all $\ell \in \mathcal{O}(n)$, there exists an n-vertex graph G with $capt(G, 1, \ell) \in \Omega(\ell \cdot n)$.*

[1] Bonato and Nowakowski also compared it to Pac-Man [7].

[2] We note that it was previously known that $capt(G, 1, 1) \leq n - 3$ for $n \geq 5$ [5].

The combination of both yields that the upper and lower bounds are asymptotically tight, i.e., the capture time of general graphs is $\Theta(\ell \cdot n)$ for $\ell \in \mathcal{O}(n)$.

Before proving Theorem 2, let us start with the example of a tree without branches, i.e., a path: While it might take the cop $D/2 \in \Omega(n)$ time to capture the first robber, all subsequent robbers can be captured in $n-1$ moves, inducing a total capture time of less than $2n$. However, the goal is to force the cop to move a linear number of times for each robber, which is shown in the following proof. Thus, we will construct a graph where, akin to a path, the robber has to move to one end to catch one robber, but then all robbers escape to the other end of the path, inducing a linear capture time for each robber. To ensure that the robbers can escape to the other side, at each end there will be a star with a ray-length of two, cf. Figure 1. Next, we will give a proof for Theorem 2.

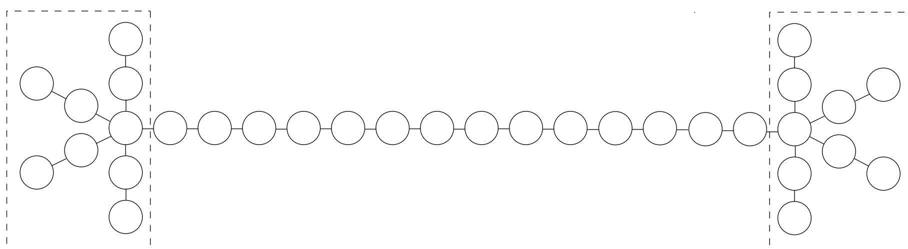

Fig. 1. Let $S_{x,y}$ be the star that has y rays of length x, i.e., the common star with n nodes would be $S_{1,n-1}$. In this example, there are two $S_{2,4}$, connected by a path of nodes. Consider a game of one cop and four robbers on this graph with 32 nodes. If in round 0 the four robbers choose a star that is farthest from the cop's initial position, and place themselves at the end of the four rays, then the cop needs at least $10 \geq n/4$ moves to capture the first robber. As the remaining robbers will flee to the other star, the cop needs $19 > n/2$ moves to catch each further robber, inducing a total capture time of 67. This construction can be directly extended to $S_{2,\lfloor n/8 \rfloor}$ for every $\ell \leq n/8$.

Proof (of Theorem 2). We begin by describing the construction of the graph G with capture time $\Omega(\ell \cdot n)$. Let $n \geq 12$ and suppose for now that $\ell \leq n/8$. Let $S_{x,y}$ be the star that has y rays of length x. Create two stars $S_{2,\ell}$, denoted by left star and right star in this proof. Connect the center nodes of both stars by a path of the remaining nodes. Note that this path (including the center of both stars) has a length (of nodes) of least $n/2 \in \Omega(n)$.

When the cop places herself in round 0, she can be either directly in the middle of the path, or closer to the left or the right star center.

Before describing the strategy of the cop, we first describe the strategy of the robbers: In round 0, all the robbers choose a star which has a higher distance to the cop than the other star (or, to break symmetry, the left one if the distance is equal); and place one robber at the end of each the ℓ rays of the star. Until the cop enters one the rays of their star, all robbers stay put. Then, when the cop is one move into the ray from the center of the star, the corresponding robber stays put, but all other robbers move one step towards the center of their star. Should the cop now move back to the center of the star, then all robbers go the end of their rays again. But if the cop moves to the end of the ray to catch a robber, then all the robbers move to the other star and choose pairwise different rays to place themselves at the end. The strategy is iterated until all robbers are caught.

We now describe a lower bound for the number of moves needed by the cop to counter the robbers' strategy: To catch the first robber, no matter where the cop starts, she has to move $\Omega(n)$ times and she has to move to the end of a ray to do so, i.e., $\text{capt}(G, 1, r_1) \in \Omega(n)$. Once a single robber is captured, all other robbers will move to the end of the rays of the other star, and the cop has no possibility to catch them before that. Thus, $\Omega(n)$ moves are needed to capture the second robber, and so on, leading to $\text{capt}(G, 1, r_\ell) = \text{capt}(G, 1, \ell) \in \Omega(\ell \cdot n)$.

Should the amount of robbers be larger than $n/8$, then both stars are created as $S_{2,\lfloor n/8 \rfloor}$, and we ignore all robbers $r_{\lfloor n/8 \rfloor+1}, \ldots, r_\ell$. Even if they should all be captured in the first round, the capture time for the remaining robbers will still be $\Omega(\ell \cdot n)$. □

Corollary 3. *For all $n \geq 12$ there exists a n-vertex graph G s.t. the number $\ell \leq n$ of robbers can be chosen with $\text{capt}(G, 1, \ell) \in \Theta(n^2)$.*

4 Many Cops, One Robber

In this section, we turn our attention to the case of many cops and extend our results for an arbitrarily large fixed number of cops:

Theorem 4. *Let $k_0 \geq 2$ be a positive integer. There exists an integer $k \geq k_0$ and a graph $G = (V, E)$ with $|V| = n \in \mathcal{O}(k^2)$ s.t. $c(G) = k$ and $\text{capt}(G, k) \in \Omega(n)$.*

In other words, we claim that the time required to catch the first robber is asymptotically linear in the number of nodes of the graph. Furthermore, we do not only prove the case that the number of cops k is a constant, we propose a stronger claim, which states that the number of nodes n our construction requires is in the order of $\mathcal{O}(k^2)$, i.e., $k \in \mathcal{O}(\sqrt{n})$.

For this effort, we utilize a graph construction by Prałat that shows the existence of graphs with n nodes with a cop number of $\Omega(\sqrt{n})$ [24].

Lemma 5. *[24] Let $c(n)$ denote the maximum of $c(G)$ over all connected graphs with n vertices. Then, $c(n) > \sqrt{n/2} - n^{0.2625}$ for n sufficiently large.*

While Lemma 5 provides us with the existence of graphs with a high cop number, the capture time in these graphs remains low, i.e., a small constant. At first sight, it is not clear if the capture time of a graph can be high in the presence of "many" cops. As a trivial example, the capture time for a single robber is 1 if the number of cops is at least $n/2$, since the initial positions of the cops can be chosen such that each node can be reached within 1 step.

However, finding an equally straightforward bound becomes more elusive when the number of cops is smaller than the size of the smallest dominating set[3] in a graph. For the case of one cop and one robber, it is known that the capture time of any 1-copwin graph is at most linear in the number of nodes. As the next step, we show a more general result showing that a similar bound holds for the case of many cops.

The basic idea is to first fix some integer k_0 and take two copies G_1 and G_2 of a graph G with $\mathcal{O}(k^2)$ nodes with cop number $k \geq k_0$ promised by Lemma 5. Then, we connect these graphs with a long *bridge* (i.e., a path) with $b \in \Omega(n)$ nodes in a similar fashion as in Section 3. The endpoints e_1 and e_2 of the bridge are connected to arbitrary nodes s_1 and s_2 in graphs G_1 and G_2, respectively. Notice that since $k \in \mathcal{O}(\sqrt{n})$, we can choose $b \in \Omega(n)$. We denote the graph constructed in this manner by $\mathcal{D}(G, b)$. See Figure 2 for an illustration.

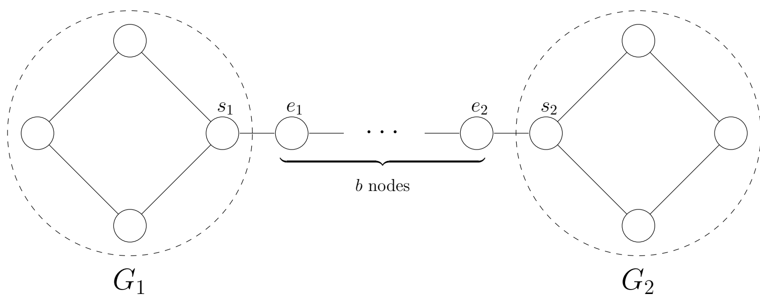

Fig. 2. Construction of the graph $D(G, b)$ for the case of two cops. The entry nodes s_1 and s_2 are chosen arbitrarily and connected by a path of length b. One cop is not sufficient to catch the robber alone in graphs G_1 and G_2.

4.1 Evasion Strategy

The next step is to choose the initial placement for the robber according to the initial placement of the cops in graph $D(G, b) = (V, E)$ for some G and b. Let $X = \{x_1, \ldots, x_k\}$, where $x_i \in V$ for every $1 \leq i \leq k$, be the initial placements of the cops, i.e., cop p_i is initially located in node x_i. In addition, let $d(u, v)$ denote the length of the shortest path between nodes u and v and $d(u, A) = \min\{d(u, v) \mid v \in A\}$, where $A \subseteq V$. We say that G_1 is *away* from the cops if

$$|\{x \in X \mid d(x, G_1) \geq \lfloor b/2 \rfloor\}| \geq \lceil k/2 \rceil ,$$

i.e., half of the cops are at most as close to G_1 as they are to G_2. Being away is defined similarly for G_2. It is easy to verify that either G_1 or G_2 is away from the cops for any k, possibly both.

[3] A set $D \subseteq V$ is a dominating set for V if every node in $V \setminus D$ has a neighbor in D.

Intuitively, we aim to locate the robber into one of graphs that is away from the cops, say G_1. By doing this, the fact that the cop number of G_1 is k ensures that the robber has a strategy that allows it to evade any number of cops trying to catch him before every cop has entered G_1. However, before stating the claim formally, there are some minor technicalities that have to be accounted first. Our construction slightly modifies graph G_1 (and G_2) by adding one additional edge that connects the bridge to G_1 and this can have an effect on the strategy of the robber that allows him to evade up to $k-1$ cops.

Fortunately, there is a simple way to show that this is not an issue. Consider now only the graph G_1 and the strategy $\mathcal{S}$ that the robber has to evade at most $k-1$ cops in G_1 indefinitely. To utilize $\mathcal{S}$ in graph $D(G,b)$, we extend it to the larger graph in the following manner. Consider any configuration of the game, i.e., the placements of the players in which there are $k' \leq k-1$ cops occupying nodes $u_1, \ldots, u_{k'}$ in G_1. Then, the robber selects the move from $\mathcal{S}$ that corresponds to a configuration in which the k' cops occupy the nodes $u_1, \ldots, u_{k'}$ in G_1 and the remaining $k-1-k'$ cops occupy node s_1.

Observation 6 *Let $G = (V, E)$ be a graph with $c(G) = k$ and let $H = (V \cup \{v\}, E \cup \{(u, v)\})$, where $v \notin V$ and u is an arbitrary node in V. Then, $c(H) \geq k$.*

4.2 The Cop Number

We have now gathered the tools that our approach requires to show that capturing one robber with many cops takes at least linear time in our graph construction. However, before going to the claim, we make one more observation about our construction. That is, we show that the cop number of $\mathcal{D}(G,b)$ is at most $c(G)+1$ and at least $c(G)$ for any b. While it might seem that adding the bridge between two graphs with cop number k should not increase the cop number, it is not clear that the robber cannot trick the cops and escape from G_1 to G_2 once all the cops have entered G_1. As a simple example of the aforementioned issue, consider the graph shown in Figure 3. While graph G in the example is a 1-copwin graph, adding an edge between two copies of G breaks this property. The 1-copwin properties are easy to verify by recalling a graph is 1-copwin if and only if it can be reduced to a single vertex by successively removing *corners*, i.e., nodes whose (inclusive) neighborhood is contained in the neighborhood of some other node [1].

We tackle this issue by observing that the cop number in the graph we have constructed is either k or $k+1$, given that the cop number of G is k. It is clearly the case that $c(\mathcal{D}(G,b)) \leq k+1$ since the game, from the perspective of the cops, can be "reduced" to playing it only in G_1 by simply leaving one cop to guard the bridge. Now the cops chasing the robber can use a strategy that does every move with the robber not in G_1 as if the robber was in s_1. If the robber decides to leave G_1, the cop guarding the bridge can capture the robber.

Lemma 7. *Let $G = (V, E)$ be a graph with $c(G) = k \geq 2$. Then, $k \leq c(\mathcal{D}(G,b)) \leq k+1$ for any integer $b > 0$.*

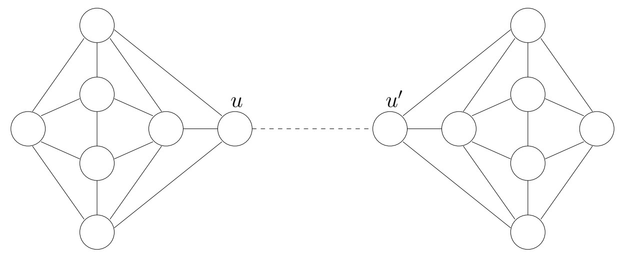

Fig. 3. Two copies of a graph G with cop number 1 connected by an edge denoted by the dashed line. Node u is the unique corner in G. Since adding the dashed edge adds a node into the neighborhood of u that is not in the neighborhood of any other node in G, it follows that u is not a corner after addition of this edge. Due to the symmetry of this example, there are no corners in the resulting construction and therefore, it is not 1-copwin.

Proof (of Lemma 7). It was shown by Berarducci and Intrigila [3] that if H is an induced subgraph of G and there exists a graph homomorphism from G onto H, which is the identity mapping in H, then $k = c(G) \geq c(H)$. Since $c(G) = k$, G is an induced subgraph of $\mathcal{D}(G, b)$ and the homomorphism can be found by considering a mapping where G_2 is mapped onto G_1 and every node in the bridge to s_1, it follows that $k \leq c(\mathcal{D}(G, b))$. Given $k + 1$ cops, one of the cops can guard the bridge and force the robber never to exit either G_1 or G_2. Therefore, the remaining k cops can simply apply the strategy promised by the fact that $c(G) = k$ to capture the robber. □

We are now ready to show that the capture time is at least asymptotically linear in the number of the nodes, for an arbitrarily large number of cops.

Proof (of Theorem 4). Let H be a graph with m nodes and a cop number of at least $\sqrt{m/2} - m^{0.2625} \geq k_0$ promised by Lemma 5. Set $G = \mathcal{D}(H, b)$, where $b = m$, and let $c(H) = k'$. Consider now the game with $c(G) = k$ cops, where $k' \leq k \leq k' + 1$ by Lemma 7, one robber, and let G_1 and G_2 be the copies of H in G. Assume without loss of generality that the cops are away from G_1. According to Observation 6, the robber has a strategy that allows him to escape at most $k' - 1$ cops as long as not every cop has entered the subgraph induced by $G_1 \cup \{e_1\}$, where e_1 is the endpoint of the bridge connected to G_1. By definition of the cops being away from G_1, there are at most $\lfloor k/2 \rfloor \geq k' - 1$ cops that are closer to G_1 than to G_2. Thus, there is at least one cop that has to move $\Omega(b) \in \Omega(n)$ times before $G_1 \cup \{e_1\}$ is occupied by at least k' cops for the first time. Since the robber is not captured before this happens and the case for G_2 works analogously, the claim follows. □

5 Many Cops, Many Robbers

Sections 3 and 4 dealt with the case of one cop and one robber, respectively. We now focus on the case of multiple cops and multiple robbers. Our goal is to show that there are k-copwin graphs for arbitrarily large k, s.t. capturing all ℓ robbers with the k cops must take in the order of $\ell \cdot b$ time, with the number of nodes in the graph being in the order of at most $\ell \cdot k^2 + \ell^2 \cdot k + b$. In particular, our goal is establish Theorem 8:

Theorem 8. *Let $k_0 \geq 2$ be a positive integer. There exists an integer $k \geq k_0$, s.t. for all $b \in \mathbb{N}$ and for all $\ell \in \mathbb{N}$ holds: There exists a graph $G_{k,\ell,b} = (V_{k,\ell,b}, E_{k,\ell,b})$ with i) $c(G_{k,\ell,b}) = k$ or $c(G_{k,\ell,b}) = k+1$ and ii) $|V_{k,\ell,b}| \in O(\ell \cdot k^2 + \ell^2 \cdot k + b)$, s.t. $capt(G_{k,\ell,b}, c(G_{k,\ell,b}), \ell) \in \Omega(\ell \cdot b)$.*

From Theorem 8, we can directly claim the following corollary, which shows that the capture time is at least asymptotically linear in the number of the nodes times the number of robbers, for an arbitrarily large number of cops and robbers by setting $b \in \Theta(\ell \cdot k^2 + \ell^2 \cdot k)$:

Corollary 9. *Let $k_0 \geq 2$ be a positive integer. There exists an integer $k \geq k_0$ and a graph $G = (V, E)$ with $|V| = n \in \mathcal{O}(\ell \cdot k^2 + \ell^2 \cdot k)$ s.t. $c(G) = k$ and $capt(G, k, \ell) \in \Omega(\ell \cdot n)$.*

We note that the cop number k can be as high as $\mathcal{O}(\sqrt{n/\ell})$, and the amount of robbers as high as $\mathcal{O}(\sqrt{n/\ k})$ in Corollary 9.

To prove Theorem 8, we cannot use a graph construction similar to the ones in Section 3 or Section 4. In Section 4, the construction relied on the fact that there is just one robber. When connecting the two copies of the graph promised by Lemma 5, the robber can pick the side with less cops – and some cops have to cross the long bride, inducing a linear capture time. However, this construction cannot be coupled with the idea of Section 3, as we cannot rule out a single cop waiting on the bridge: Connecting the graphs promised by Lemma 5 can increase the cop number by one (cf. Figure 3). Then, the robbers cannot escape over the bridge, allowing the cops to capture them in rapid succession. Even if one would add multiple bridges, the extra cop could simulate the behavior of the robbers, capturing at least a fraction of them each time the robbers cross. Thus, we need an improved graph family to establish Theorem 8.

In the following, we first describe the new graph construction (Subsection 5.1) with the desired properties and the strategy of the robbers in these graphs (Subsection 5.2), before we prove Theorem 8 in Subsection 5.3.

5.1 The Graph Construction of $G_{k,\ell,b}$

Given an integer k_0, Lemma 5 promises a graph $G_g = (V_g, E_g)$ with a cop number of $c(G_g) = k \geq k_0$ and at most $|V_g| \in O(k^2)$ nodes. Henceforth, we will refer to these graphs G_g as *gadget graphs* with a diameter of $D(G_g) = D_g$.

The construction idea of this subsection is as follows: We construct a cycle and attach ℓ copies of the gadget graph with long lines to top, bottom, left, and right side of the cycle. A graphical depiction can be found in Figure 4.

We first describe how to attach the copies: Let v_g be a fixed node in G_g. Attach a line of nodes of length $10\ell \cdot D_g$ to v_g. Then, copy the graph G_g with

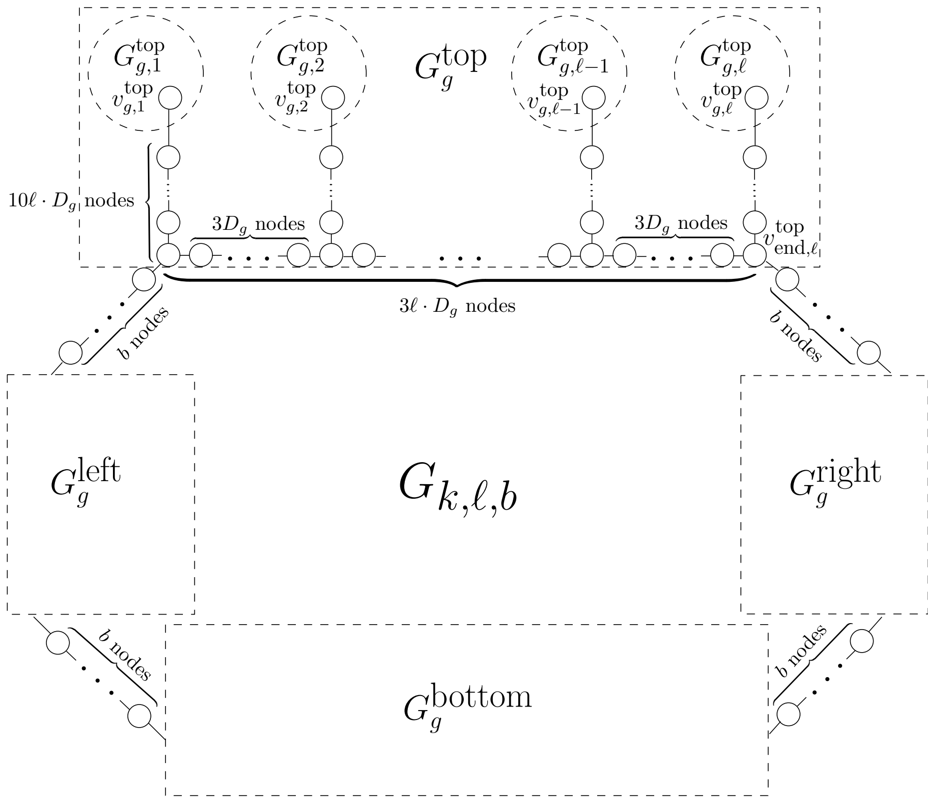

Fig. 4. The graph $G_{k,\ell,b}$ consists of the four subgraphs G_g^{top}, G_g^{bottom}, G_g^{left}, and G_g^{right}, which are connected cyclically by paths of length b. Each of the four subgraphs consists of ℓ copies of the gadget graph G_g with $c(G_g) = k$, which is in turn connected to a path of length $10\ell \cdot D_g$ (with D_g being the diameter of G_g). Initially, the ℓ robbers place themselves all in a subgraph s.t. at least half the cops are at least $b/2$ moves away from the subgraph, one robber in each of the ℓ gadget graphs. As each robber can evade less than k cops in his gadget graph indefinitely, no robber can be caught until k cops enter his gadget graph. I.e., the cops need $\Omega(b)$ moves to capture the first robber. If there are just $G_{k,\ell,b} = k$ cops, the robbers could then all escape to another subgraph, forcing the cops to spend at least $\Omega(b)$ moves for each subsequent robber. However, if there are $c(G_{k,\ell,b}) = k + 1$ cops, the extra cop p_{k+1} could patrol anywhere in the graph, possible blocking the path of the robbers. However, if all robbers always try to move to their respective node v_{end}, but moving back when a cop comes closer than D_g, the extra cop can keep at most one extra robber in check. Then, as soon as k cops enter a gadget graph with a robber, the other robbers can escape to another subgraph: In the top subgraph case, the robbers "left" of p_{k+1} go to G_g^{left}, the robbers "right" of p_{k+1} go to G_g^{right}. As all these robbers always keep a distance of at least D_g to the next cop when escaping to another subgraph, they can position themselves perfectly in their new gadget graph with a diameter of D_g. This can be iterated, enforcing a capture time of $\Omega(\ell \cdot b)$. If $b \in \Omega(n)$ is chosen, then this yield a lower bound of $\Omega(\ell \cdot n)$. We note that three subgraphs would also suffice with a slightly modified strategy.

the line ℓ times as $G_{g,1}, \dots, G_{g,\ell}$, with the other endpoints of the lines called $v_{\text{end},1}, \dots, v_{\text{end},\ell}$ respectively. Connect $v_{\text{end},i}$ to $v_{\text{end},i+1}$ with a line of length $3 \cdot D_g$ for all $1 \le i < \ell$, inducing a path from $v_{\text{end},1}$ to $v_{\text{end},\ell}$ consisting of $3(\ell-1) \cdot D_g$ new nodes. Denote this construction as G_g^{top} and copy it three more times as G_g^{bottom}, G_g^{left}, and G_g^{right}.

Lastly, we connect all four structures with a cycle by adding $4 \cdot b$ nodes: Connect $v_{\text{end},\ell}^{\text{top}}$ by a line of length b with $v_{\text{end},1}^{\text{right}}$, $v_{\text{end},\ell}^{\text{right}}$ by a line of length b with $v_{\text{end},1}^{\text{bottom}}$, $v_{\text{end},\ell}^{\text{bottom}}$ by a line of length b with $v_{\text{end},1}^{\text{left}}$, and $v_{\text{end},\ell}^{\text{left}}$ by a line of length b with $v_{\text{end},1}^{\text{top}}$.

Lemma 10. *The graph $G_{k,\ell,b}$ has $O(\ell \cdot k^2 + \ell^2 \cdot k + b)$ nodes.*

Proof (of Lemma 10). Each of the four graphs G_g^{top}, G_g^{bottom}, G_g^{left}, and G_g^{right} consists of ℓ copies of G_g with a line of $10\ell \cdot D_g$ nodes and $3(\ell-1) \cdot D_g$ further nodes connecting them. Together with the $4b$ nodes acting as bridges, the total node count is in $G_{k,\ell,b}$ is $4\,(\ell(\cdot|V_g| + 10\ell \cdot D_g) + 3(\ell-1) \cdot D_g + b)$. Due to Lemma 5, $|V_g| \in O(k^2)$, which results in an upper bound of $O(\ell \cdot k^2 + \ell^2 \cdot D_g + b)$ nodes.

The graph construction of Lemma 5 uses $k+1$-regular graphs with $2(k^2+k+1)$ nodes [24]. As shown by, e.g., Moon in [20], the diameter D_g of G_g is therefore in $O(\frac{2(k^2+k+1)}{k+1}) \in O(k)$. Hence, the number of nodes in $G_{k,\ell,b}$ is $O(\ell \cdot k^2 + \ell^2 \cdot k + b)$. ☐

We now show that the cop number of the whole construction is at most the cop number of the gadget graph plus one:

Lemma 11. *Let G_g with $c(G_g) = k$ be the gadget graph used in the construction of $G_{k,\ell,b}$. The cop number of $G_{k,\ell,b}$ is k or $k+1$.*

Proof (of Lemma 11). The cop number of $G_{k,\ell,b}$ is at least k, as the cop number of the gadget graph G_g is already k: A robber could place themselves into a copy of G_g and just simulate his evasion strategy accordingly, with never leaving G_g.

Furthermore, $k+1$ cops suffice for $G_{k,\ell,b}$: Already two cops can force a robber to place himself into a gadget graph. Then, one cop waits at the exit node v_g, while the remaining k cops capture the robber, simulating their winning strategy from the gadget graph G_g with $c(G_g) = k$. ☐

5.2 The Robber Strategy

The robber strategy in $G_{k,\ell,b}$ can be summarized as follows: Start in the part of the graph with the fewest cops (each robber in a distinct gadget graph), then try to wait at the end of the line of the current gadget graph, only going back into the gadget graph if a cop comes close. If a cop comes into the current gadget graph, simulate an evasion strategy, which will work for sure until at least k cops enter. Then, as soon as k cops are close to any gadget graph in the subgraph, the other robbers escape to another subgraph without cops, and repeat the initial strategy. If there are $k+1$ cops in the graph, then the cop p_{k+1} may hold back one extra robber from escaping, but all the other robbers can move away from p_{k+1} to another subgraph (possible splitting the robbers into different subgraphs).

We now describe the strategy in detail: After the cops placed themselves, the robbers choose a subgraph G_g^{top}, G_g^{bottom}, G_g^{left}, G_g^{right} to start in that has the most cops being in a distance of at least $b/2$. W.l.o.g., let this subgraph be G_g^{top}. Note that due to the pigeonhole principle, at most half of the cops can be within a distance of $b/2$ near G_g^{top} and that at least k cops are needed to catch a robber in a gadget graph (cf. Lemma 11). Hence, the remaining cops need at least $b/2 + 10\ell \cdot D_G$ moves to reach any gadget graph in G_g^{top}

Each of the ℓ robbers will place themselves into a pairwise distinct gadget graph $G_{g,1}^{\text{top}}, \ldots, G_{g,\ell}^{\text{top}}$ as follows: Each robber r_i will assume that there are $k-1$ cops in his gadget graph $G_{g,i}^{\text{top}}$, with the missing ones placed all at $v_{g,i}^{\text{top}}$. Then, his placement will be identical as in his evasion strategy for the graph G_g.

Next, as soon as the distance to the nearest (real) cop is larger than D_g, the robber will move towards the node $v_{\text{end},i}^{\text{top}}$, but not surpassing it yet. Should then a cop come closer, then the robber will move back towards $G_{g,i}$, keeping a distance of at least D_g, but move forward again if the cop is further away again. When the robber enters his graph $G_{g,i}^{\text{top}}$ again with a cop close, he resumes simulating the evasion strategy until the distance to the next cop is more than D_g. The distance of D_g is necessary for the robber to assume an arbitrary starting position in the gadget graph again before the first cop enters the gadget graph.

As soon as at least k cops are in a distance of at most D_g to one of the gadget graphs at the top (the robber in this graph now stops moving), there can be at most one other cop, say p_{k+1}, left in the graph. This cop p_{k+1} can be in distance of at most D_g for only one robber (this robber now stops moving as well), allowing all other robbers R' to be at their node $v_{\text{end},i}^{\text{top}}$.

Let $G_{g,j}^{\text{top}}$ be the gadget graph to which the cop p_{k+1} is closest. Due to the construction of $G_{k,\ell,b}$, each robber in R' has now a distance of more than D_g to p_{k+1} (if it exists), and a distance of at least $9\ell \cdot D_g$ to all other cops.

Should $c(G_{k,\ell,b}) = k$, then there is no cop p_{k+1}, and all robbers from R' move to the same of one the other three subgraphs G_g^{bottom}, G_g^{left}, G_g^{right} and place themselves in pairwise distinct gadget graphs, repeating their initial strategy accordingly.

If $c(G_{k,\ell,b}) = k+1$, then the cop p_{k+1} can be closest to the node 1) $v_{\text{end},1}^{\text{top}}$, 2) $v_{\text{end},\ell}^{\text{top}}$, or 3) to some node $v_{\text{end},j}^{\text{top}}$, with $1 < j < \ell$. In the case of 1) (i.e., the cop is at the "left end"), all robbers from R' move to pairwise distinct gadget graphs in G_g^{right}. In the case of 2) (i.e., the cop is at the "right end"), all robbers from R' move to pairwise distinct gadget graphs in G_g^{left}. For the last case of 3), let $R'_<$ be the robbers of R' be at nodes $v_{\text{end},i}^{\text{top}}$ with $i < j$ and $R'_>$ be the robbers of R' be at nodes $v_{\text{end},i}^{\text{top}}$ with $i > j$. All robbers from $R'_<$ move to unoccupied pairwise distinct gadget graphs in G_g^{left}, all robbers from $R'_>$ do the same in G_g^{right}.

Afterwards, the robbers repeat their strategy, adjusted to being in G_g^{top}, G_g^{bottom}, G_g^{left}, G_g^{right} accordingly. Note that the robbers may be split up between all four subgraphs, but that it takes always at least k cops to force them to move to another subgraph.

5.3 A Lower Bound for the Capture Time

In this subsection, we will complete the proof of Theorem 8 by showing a lower bound of $\Omega(\ell \cdot b)$ on the capture time when the robbers use the strategy described in Subsection 5.2 in the graph $G_{k,l,b}$. Essentially, the cops need to move at least b times to capture a constant number of robbers, forcing a lower bound of $\Omega(\ell \cdot b)$.

Proof (of Theorem 8). With Lemma 10 (the size of the graph) and Lemma 11 (the cop number of the graph), all that is left to show of Theorem 8 is a capture time of $capt(G_{k,\ell,b}, c(G_{k,\ell,b}), \ell) \in \Omega(\ell \cdot b)$.

After the cops place themselves initially in the graph $G_{k,l,b}$, the strategy of the robbers will ensure that at least half the cops are at least $b/2$ moves away from each robber. As the robbers are initially in the gadget graphs, at least k cops are required to capture any robber in its gadget graph, requiring at least $b/2$ moves from some cops to capture the first robber.

We begin with the case of $c(G_{k,l,b}) = k$ before discussing $c(G_{k,l,b}) = k + 1$. Let w.l.o.g. G_g^{top} be the subgraph where k cops are for the first time within a distance of D_g to a gadget graph. Then, when the other robbers R' in G_g^{top} escape to another subgraph G_g^{bottom}, G_g^{left}, G_g^{right}, these robbers need at most $3\ell \cdot D_g$ moves to exit the subgraph G_g^{top}. However, each of the k cops needs at least $9\ell \cdot D_g$ moves to reach the first node of the type $v_{\text{end}}^{\text{top}}$, ensuring that the other robbers have at least a distance of D_g at all times to these k cops before they enter their new gadget graph to hide in. For every next robber to be captured, these k cops need to move thus to the next gadget graph in another subgraph, enforcing at least b moves for the cops, ensuring a capture time of $\Omega(\ell \cdot b)$.

The case of $c(G_{k,l,b}) = k + 1$ is similar, but now there is an additional cop (w.l.o.g. p_{k+1}) that might not need to enter the gadget graphs and is free to move around through the graph, possibly capturing or blocking the other robbers R'. Still, if not at least k cops enter a gadget graph at some point, no robber can be caught, as the robbers can always evade less than k cops in their gadget graph. Consider the move when at least k cops are within distance D_g to a gadget graph. Due to the strategy of the robbers, the remaining cop p_{k+1} can be within distance of D_g to at most one robber in G_g^{top}. Thus, all other robbers, which are at nodes of the type $v_{\text{end}}^{\text{top}}$, can escape to the other subgraphs (G_g^{bottom}, G_g^{left}, G_g^{right}), depending on where p_{k+1} is located – the "ring" structure of $G_{k,\ell,b}$ does not allow p_{k+1} to block the other robbers.

I.e., at most two robbers can be prevented from escaping to another subgraph. When the robbers arrive in the pairwise distinct gadget graphs of their new subgraph G_g^{bottom}, G_g^{left}, G_g^{right}, the initial situation occurs again: The cops need to have moved at least b times to capture again at most two robbers, inducing a total capture time of $\Omega(\ell \cdot b)$. □

5.4 A Superlinear Number of Robbers

So far, the number of robbers has never exceeded a linear amount, i.e., we did not consider the case of $\omega(n)$ robbers in n-vertex graphs. However, our results can be extended to this case.

We start with the case of one cop and many robbers (cf. Section 3). Fix a number of nodes n for Corollary 3 and let ℓ' be any number of robbers less than $n/8$. If the number of robbers were to be increased to $2\ell'$, then one could always move two robbers as if they were one, with them sharing the same place. Then the capture time would remain the same, as the cop would always capture two robbers at once.

However, after the first two robbers are captured, and all robbers move along the bridge connecting the left and the right star (see Figure 1), the only unoccupied ray of the star can be now be occupied by splitting a pair of robbers into singles. The cop could still capture two robbers in his next catch, but only $\ell'/2$ times in total! After that, all rays would only be occupied by one robber, allowing the cop to capture only one robber at once. Hence, the cop now needs to cross the bridge connecting the two stars at least $\ell'/2 + \ell'$ times.

This concept can be iterated, e.g., for $4\ell'$ robbers, the cop needs to cross the bridge $(1/4 + 1/3 + 1/2 + 1)\ell'$ times. I.e., for $t \cdot \ell'$ robbers, this number increases to $(1/t + \cdots + 1/2 + 1)\ell' \in \Omega(\ell' \log t)$. With the bridge having a length of $\Omega(n)$ nodes, the following corollary holds:

Corollary 12. *For all $n \geq 12$ there exists a 1-copwin graph G s.t. for all numbers $\ell \geq n$ of robbers capt$(G, 1, \ell) \in \Omega\left(n^2 \cdot \log\left(\ell/n\right)\right)$.*

We note that a similar line of thought can be applied to the case of more than one cop, i.e., letting the cops capture multiple robbers at once, and then splitting up the remaining robbers evenly among the gadget graphs.

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