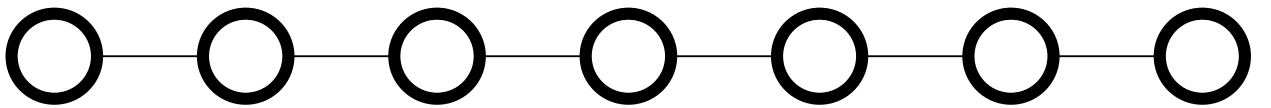
Lower Bounds for the Capture Time: Linear, Quadratic, and Beyond

Klaus-Tycho Förster, Rijad Nuridini, Jara Uitto, Roger Wattenhofer

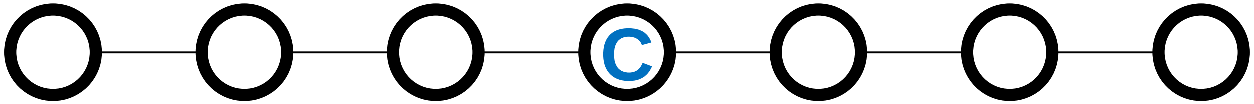
The game of Cops and Robbers

How to catch a robber on a graph?

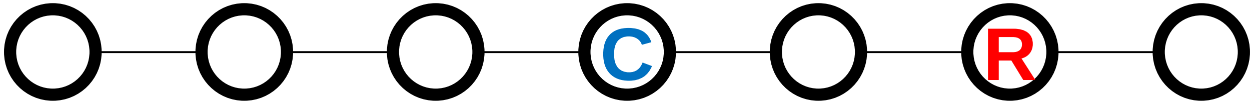
The rules of the game



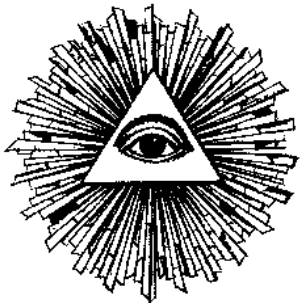
The **C**op is placed **first**

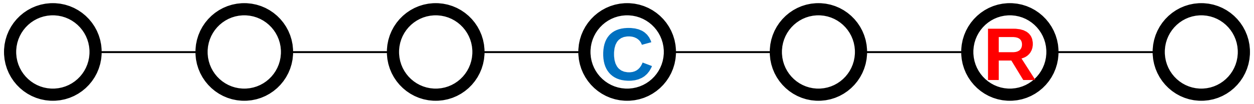


The **R**obber may **then** choose a placement



Next, they **alternate** in moves





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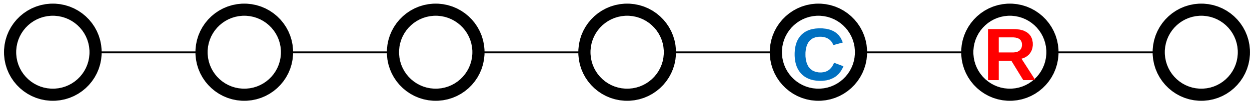

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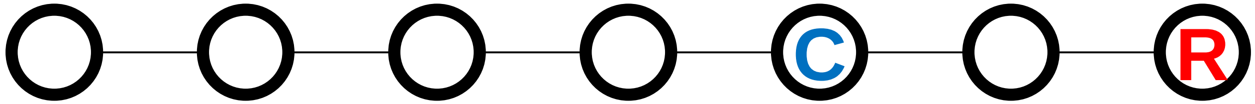

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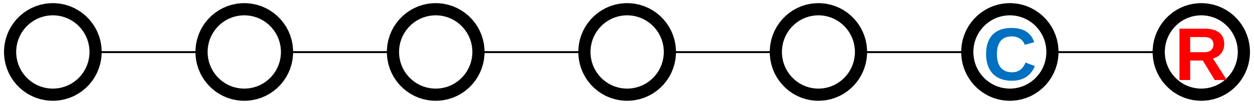
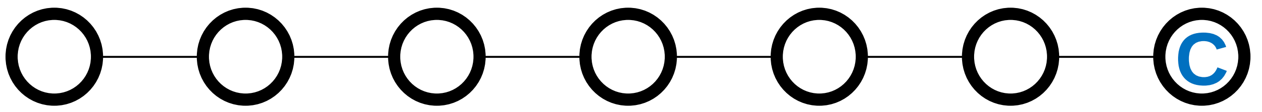


The Cop won!



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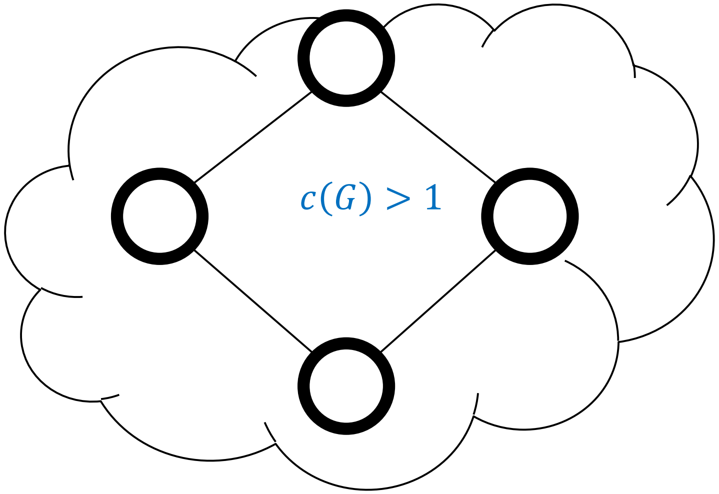
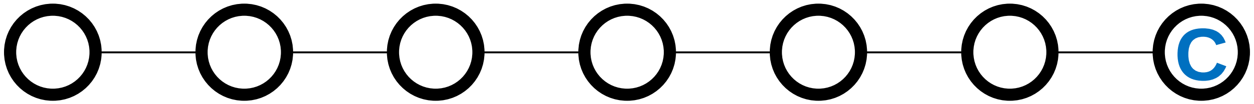
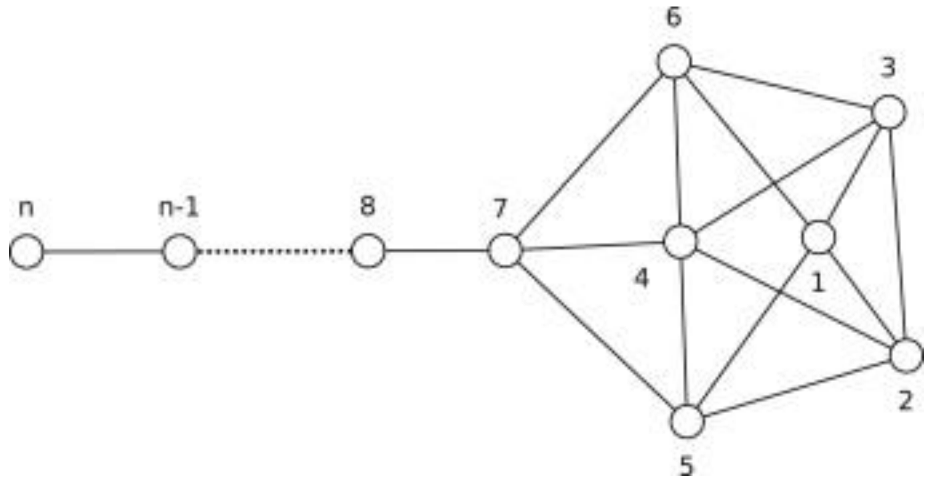




Graphs G where 1 cop wins have a **cop number** of $c(G) = 1$

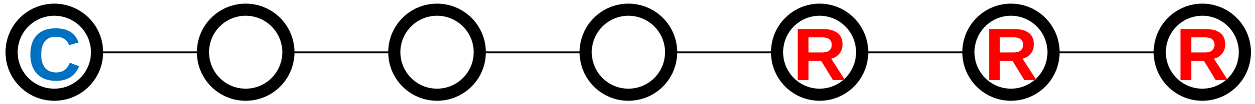
How many moves does the cop need?

- For graphs with $c(G) = 1$:
  - $\mathrm{n} \geq 7$ nodes: $\boldsymbol{n-4}$ moves always suffice
  - $\exists$ graphs where $\boldsymbol{n-4}$ moves are needed



(Gavenčiak, 2010)

Catch **multiple**?



n moves suffice for paths

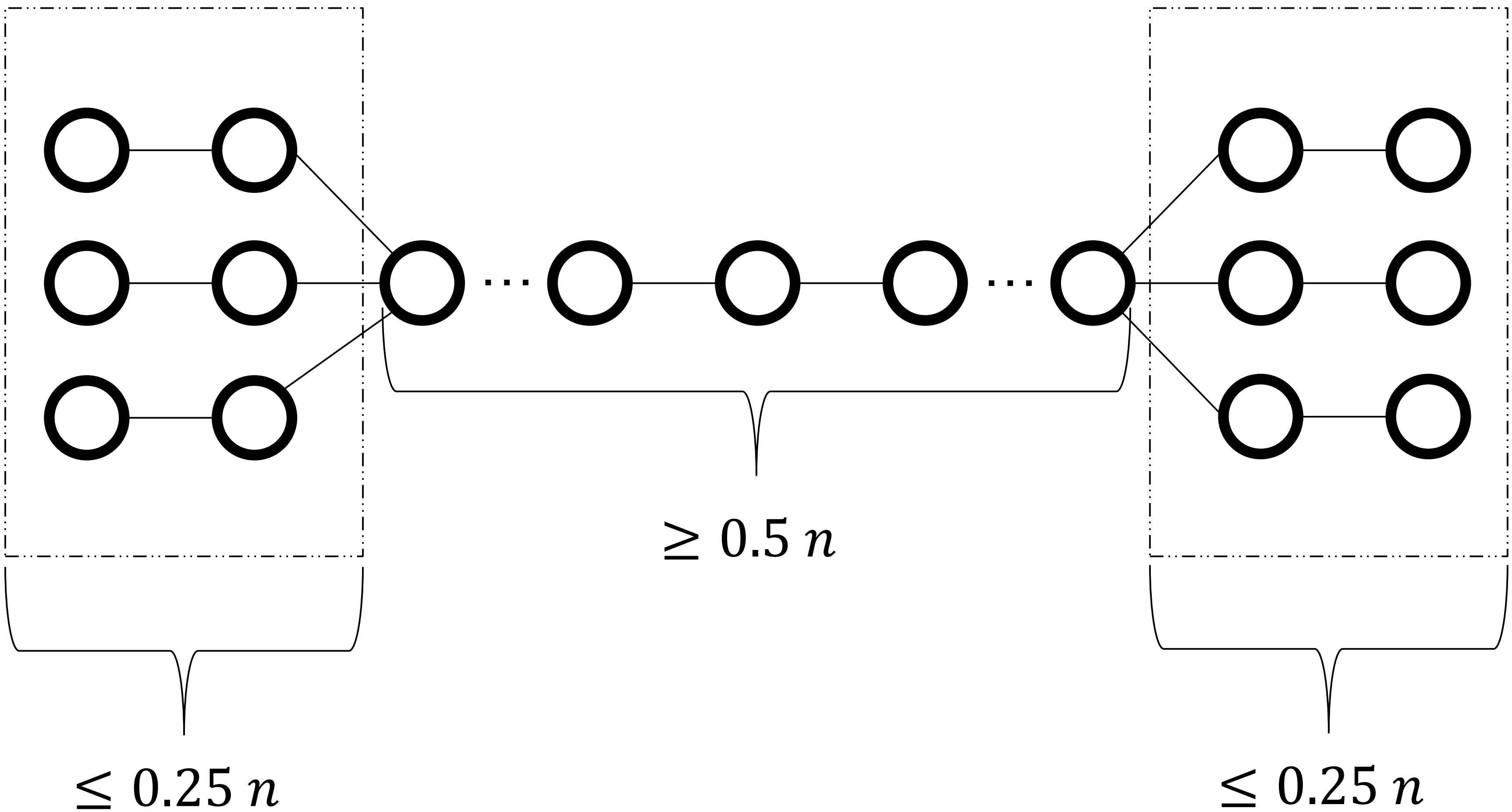
Upper bound to catch ℓ robbers

1. $n - 4$ moves for the first robber

2. Every further robber:
   - Cop moves to start in at most diameter D moves
   - $n - 4$ moves for the next robber

$\rightarrow$ $O(\ell * n)$ moves in total

Lower bound to catch ℓ robbers



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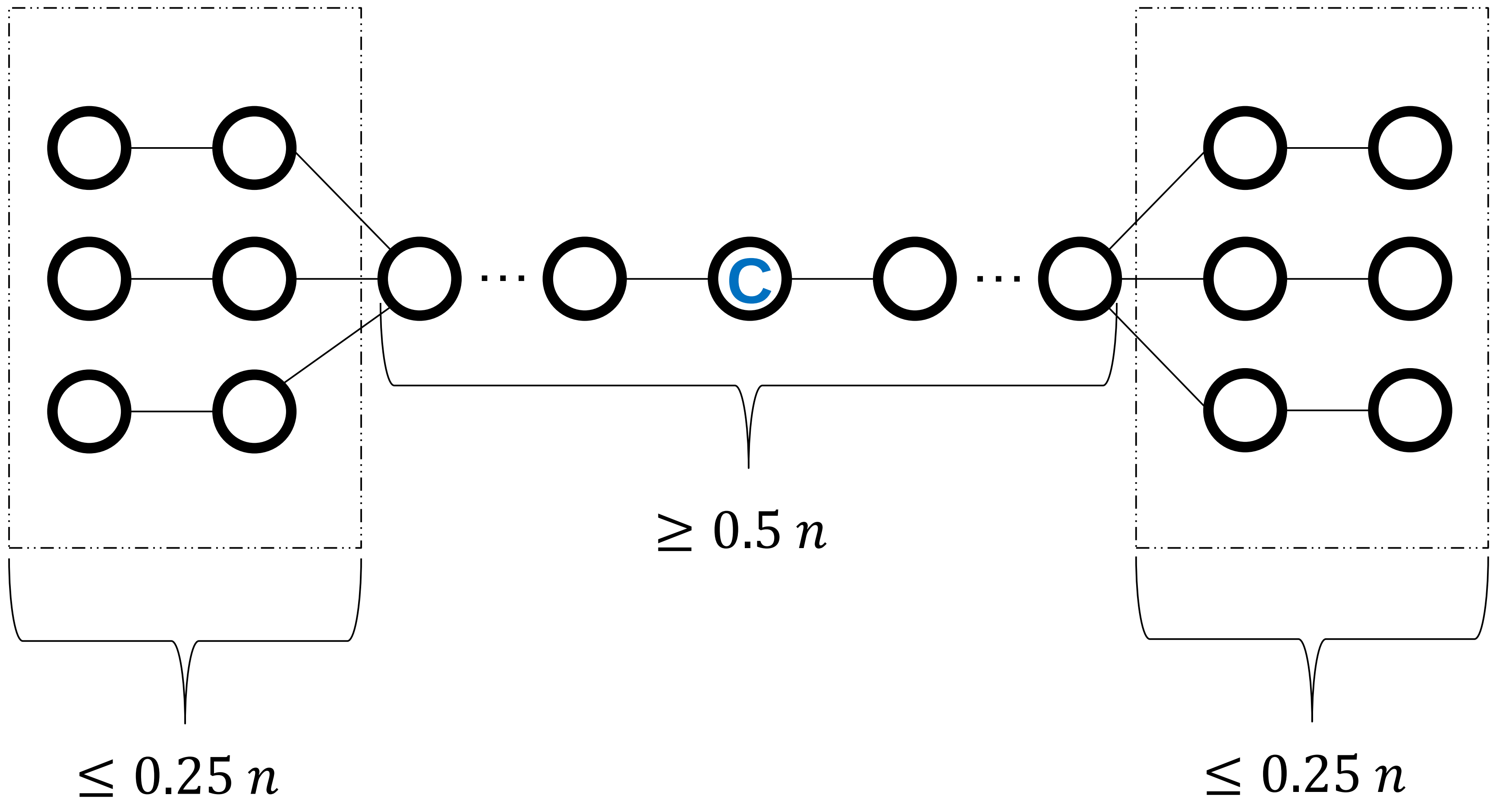
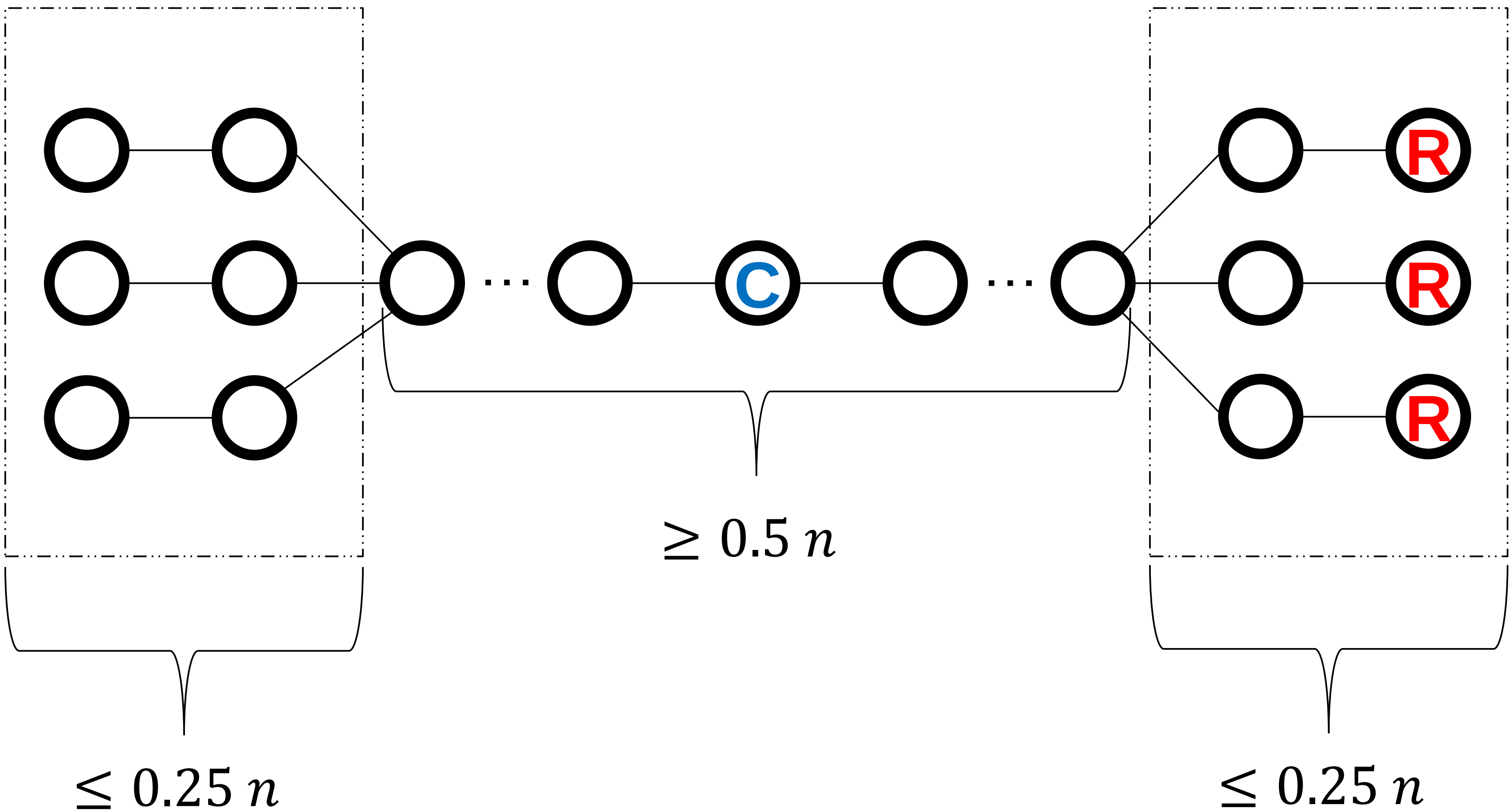


Lower bound to catch ℓ robbers



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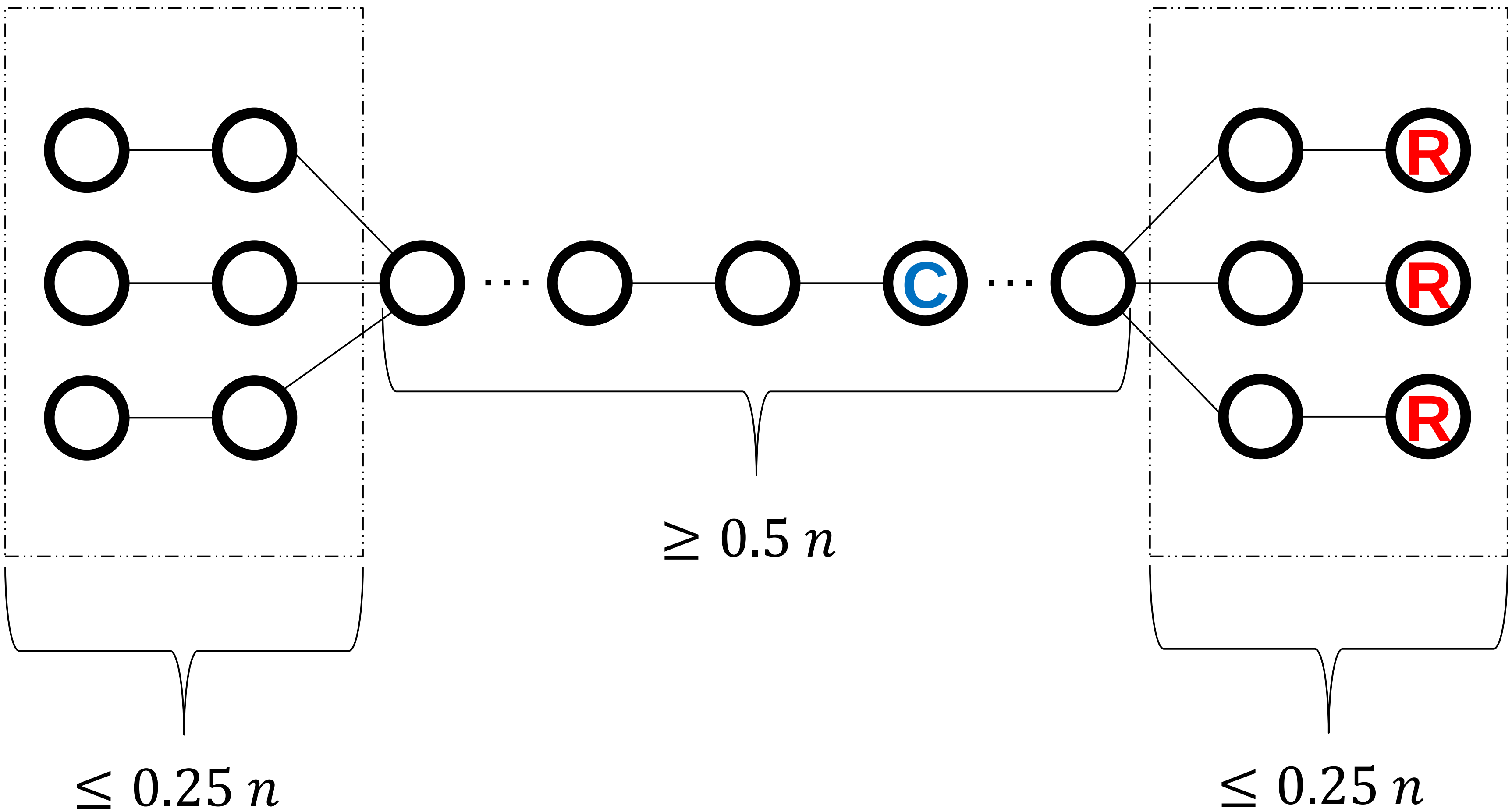

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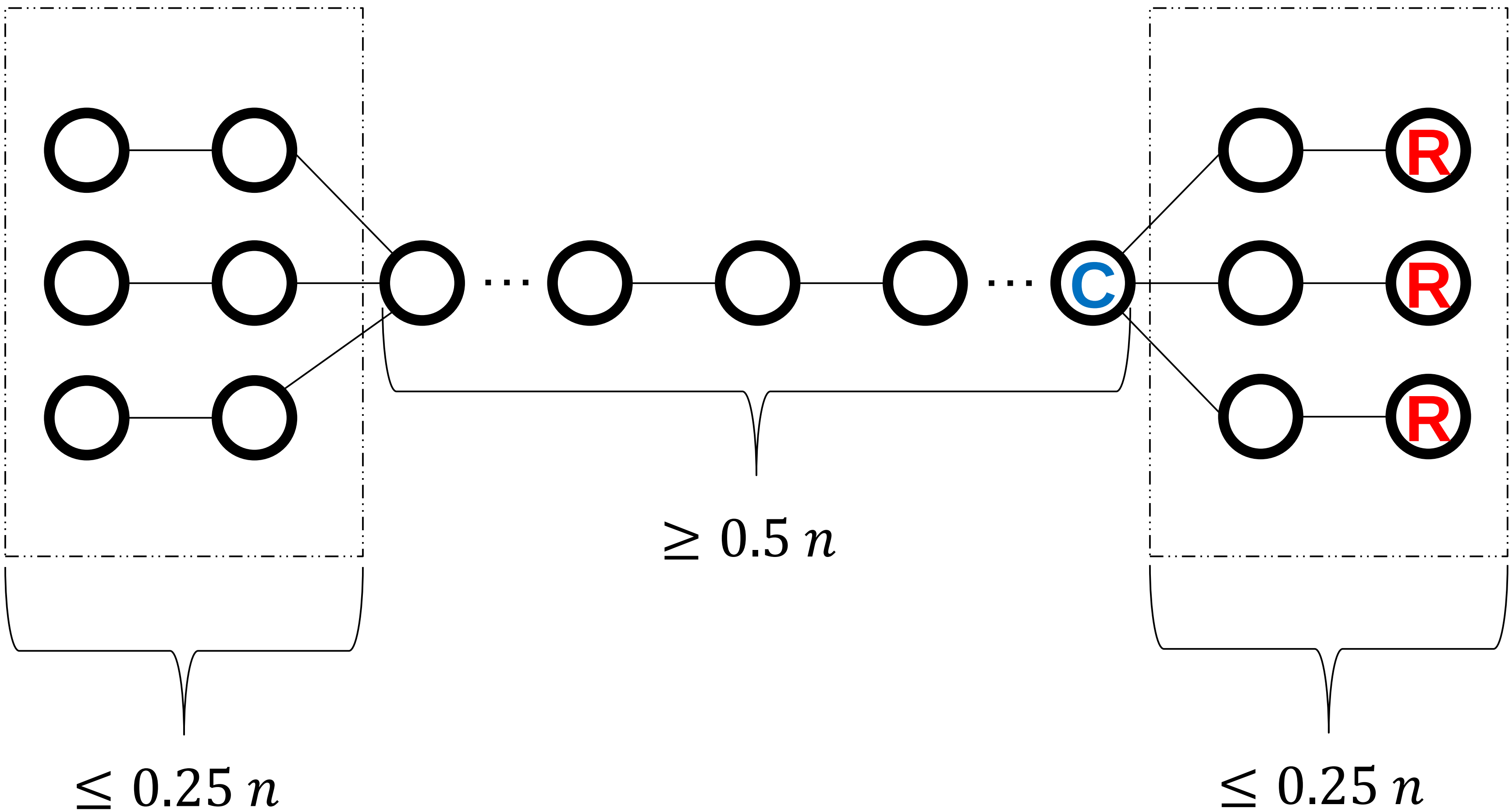

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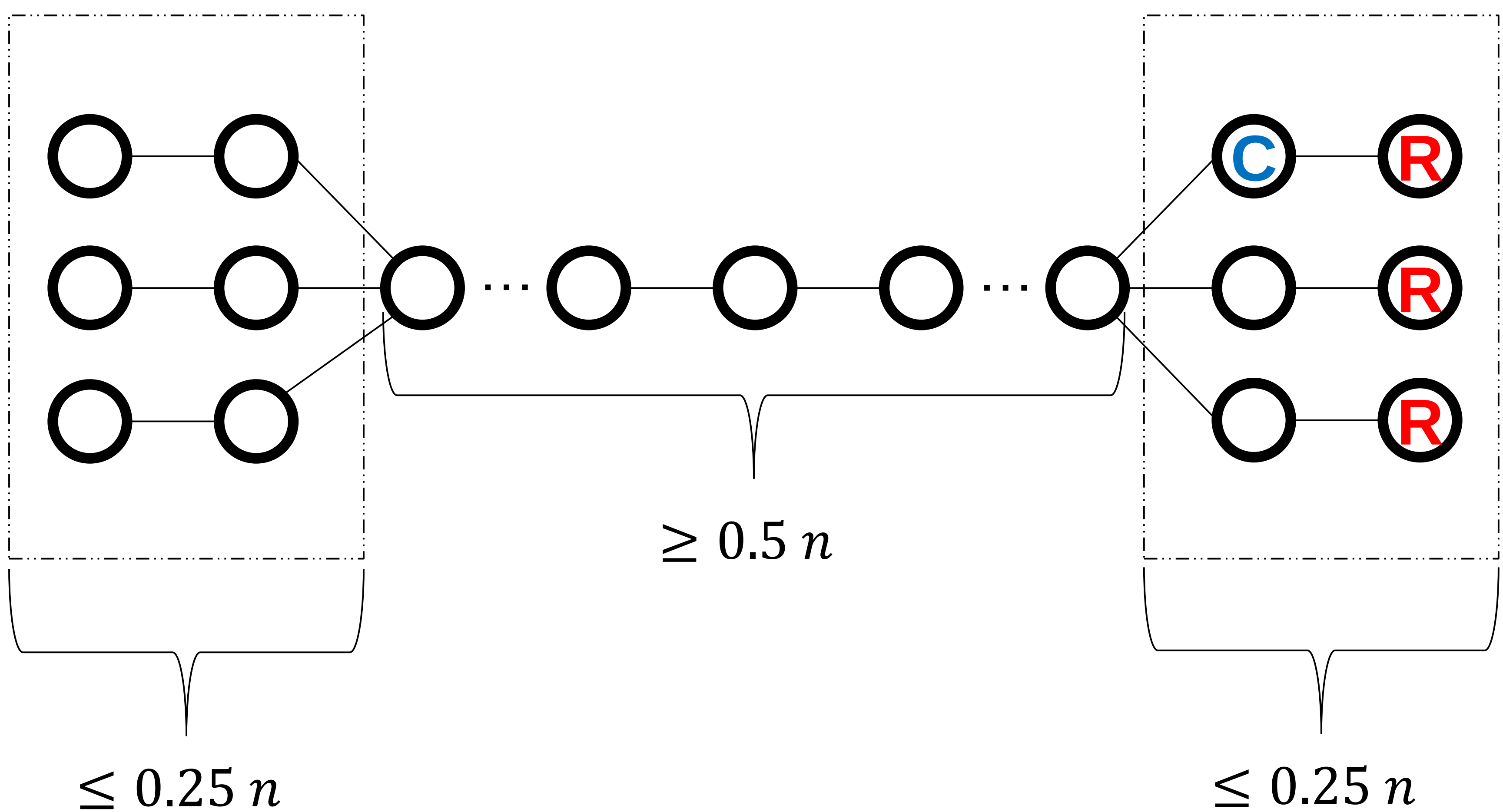

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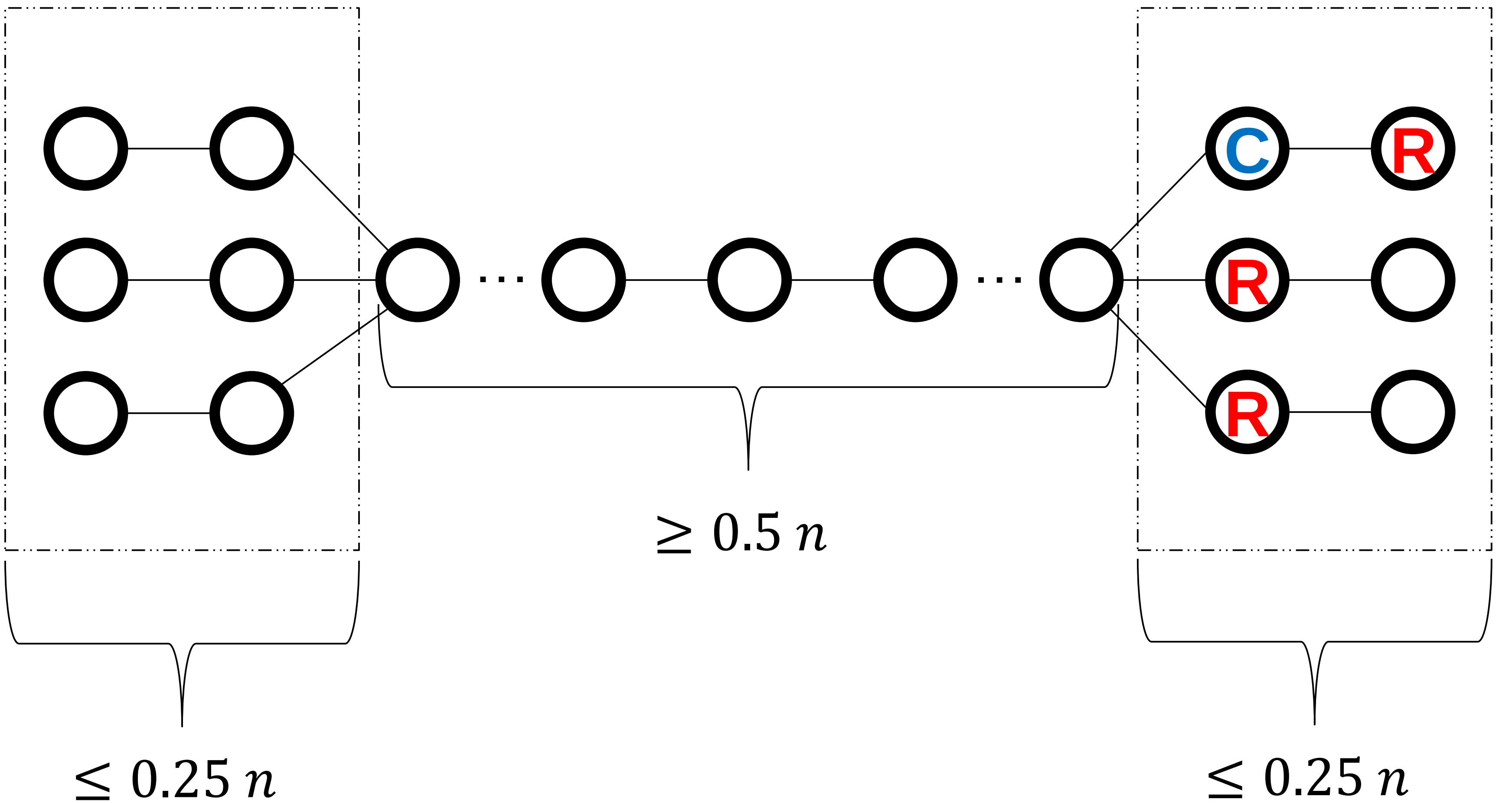

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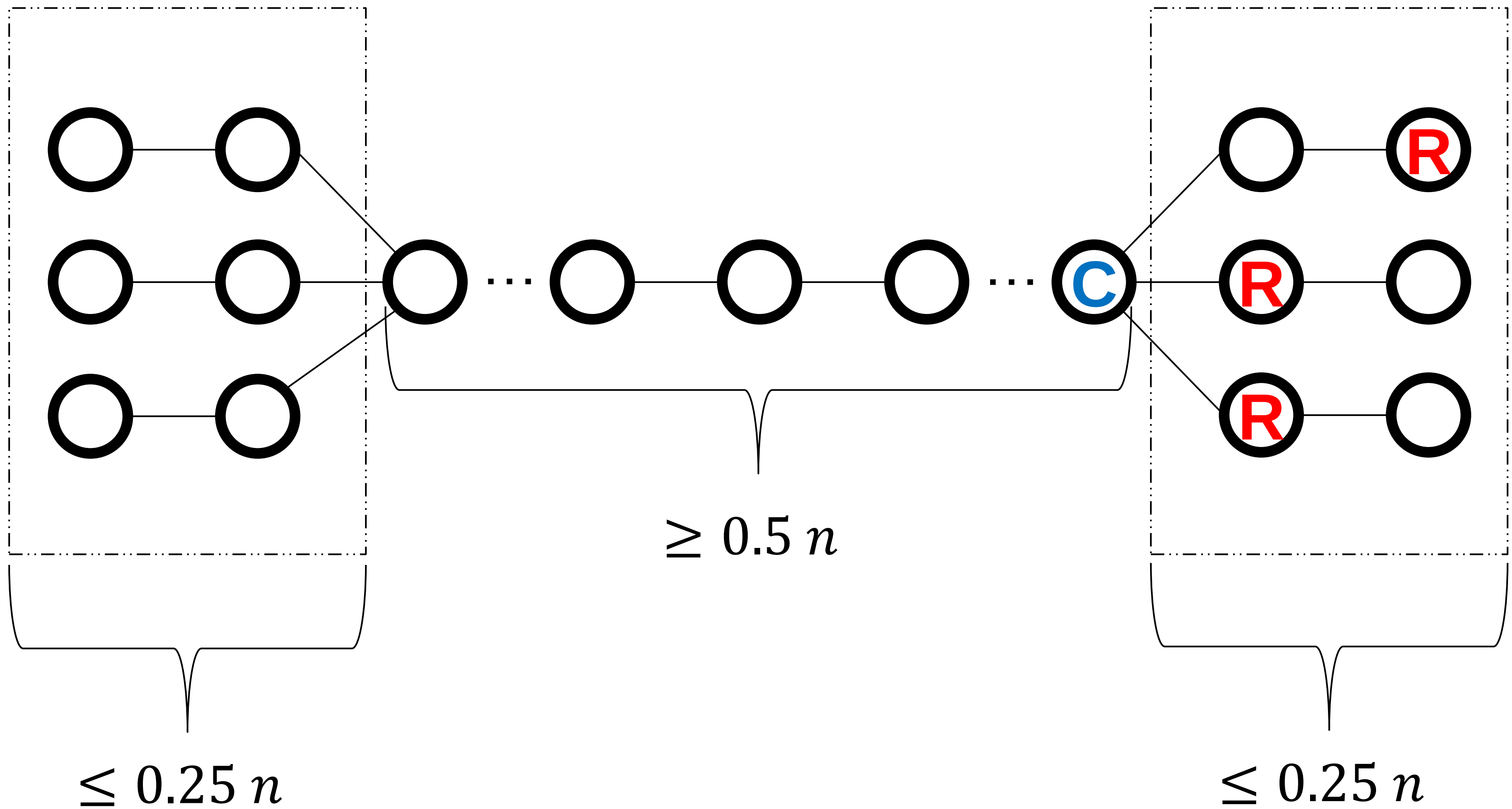

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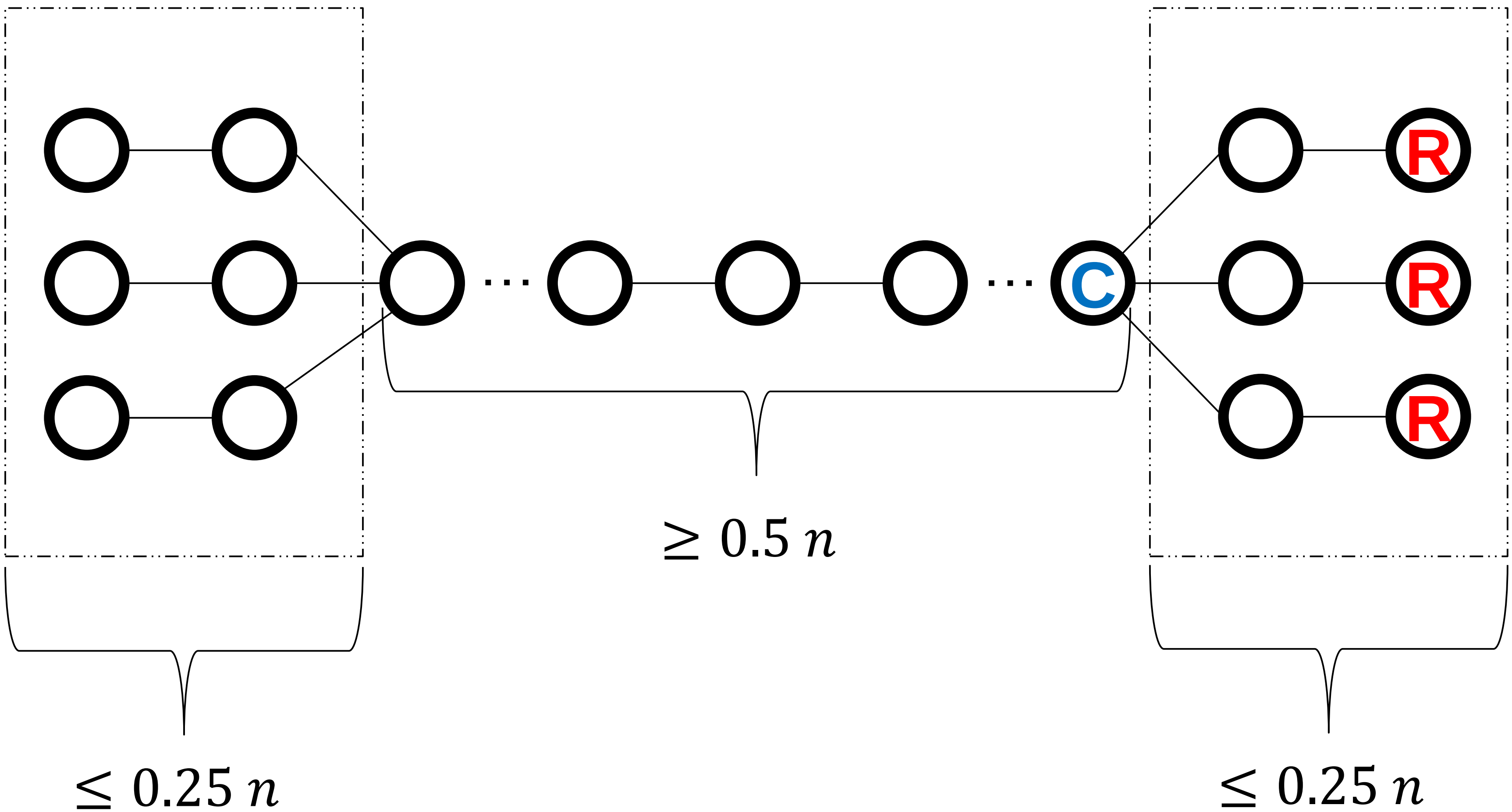

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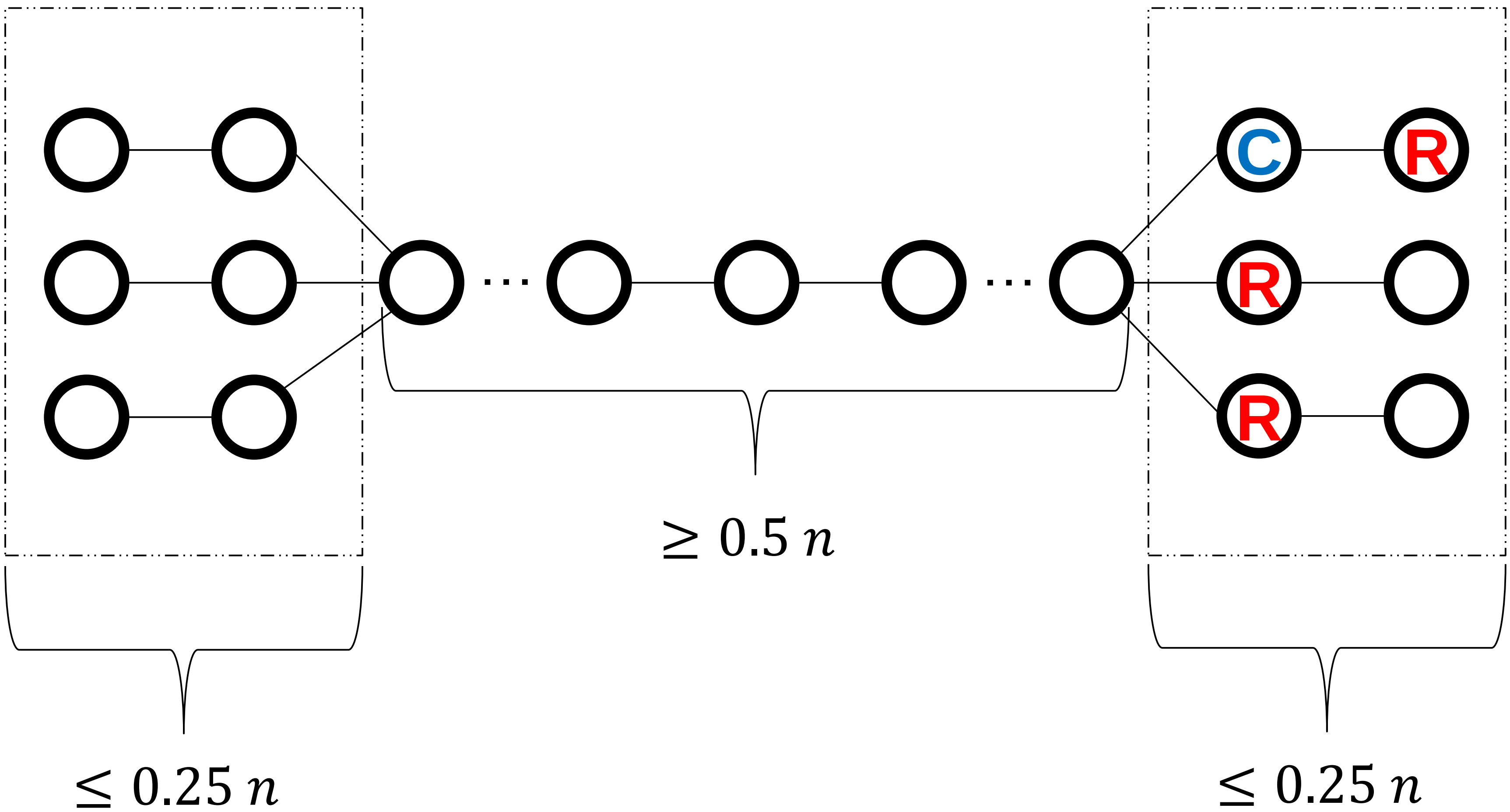

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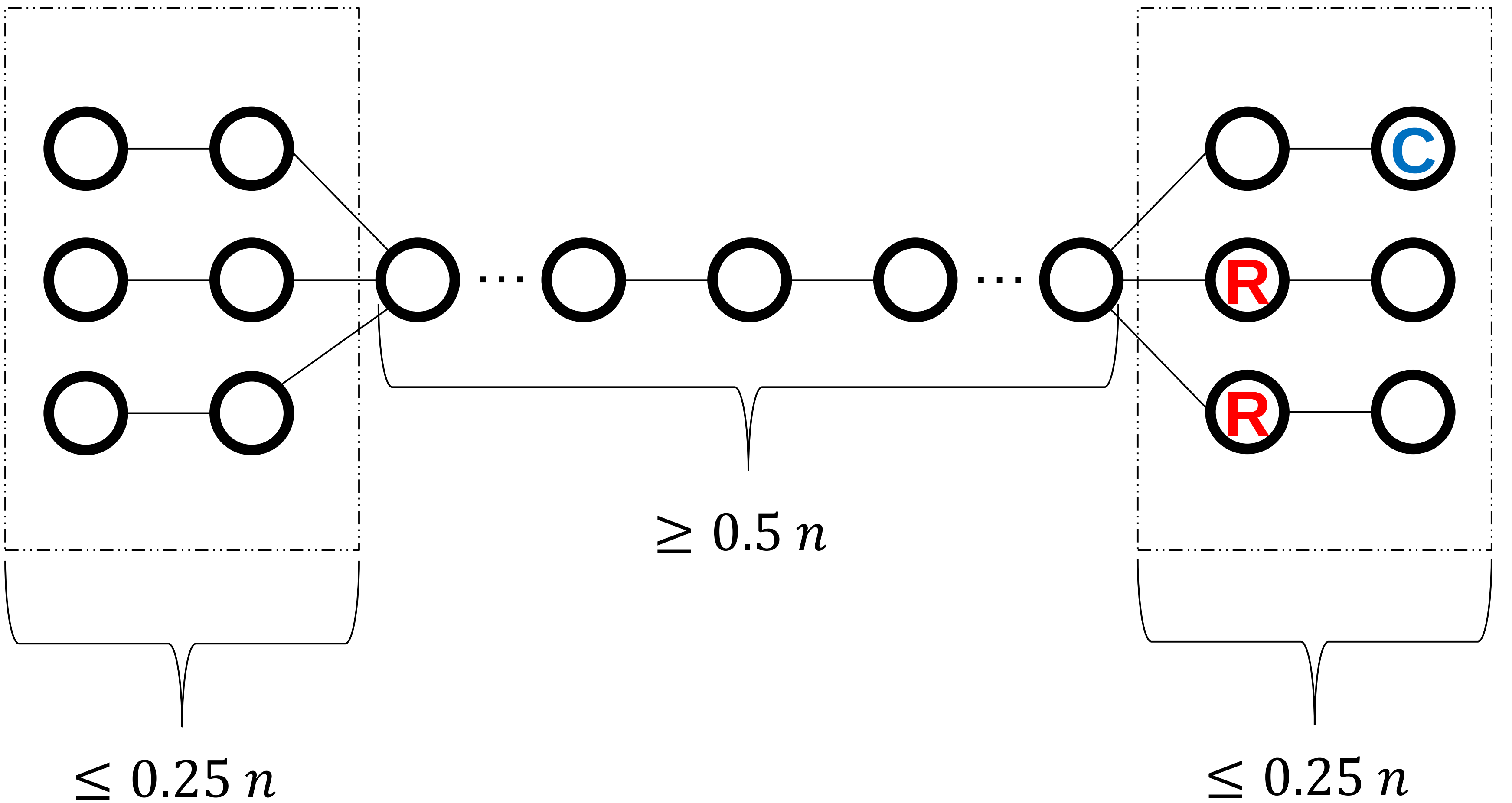

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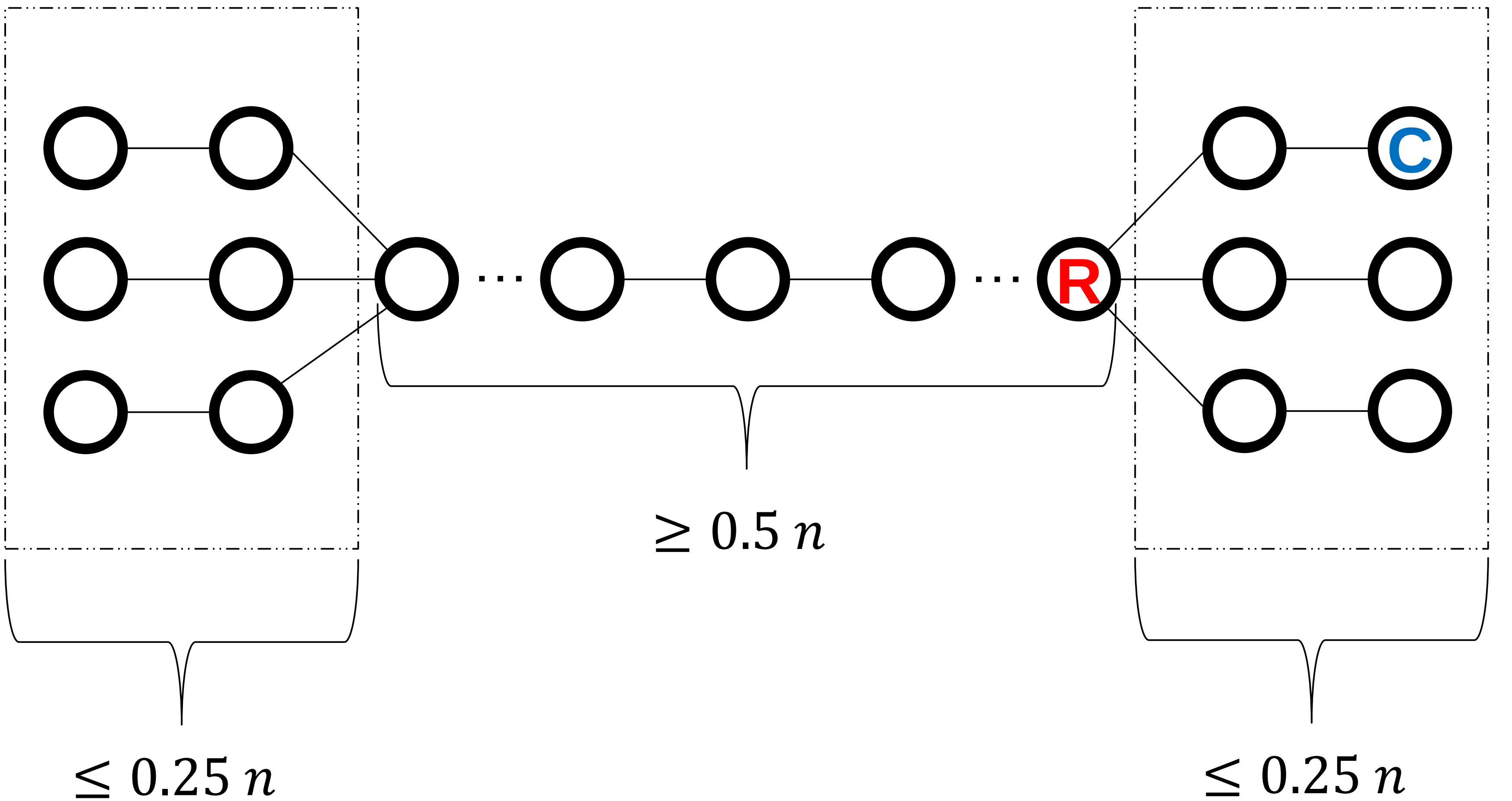

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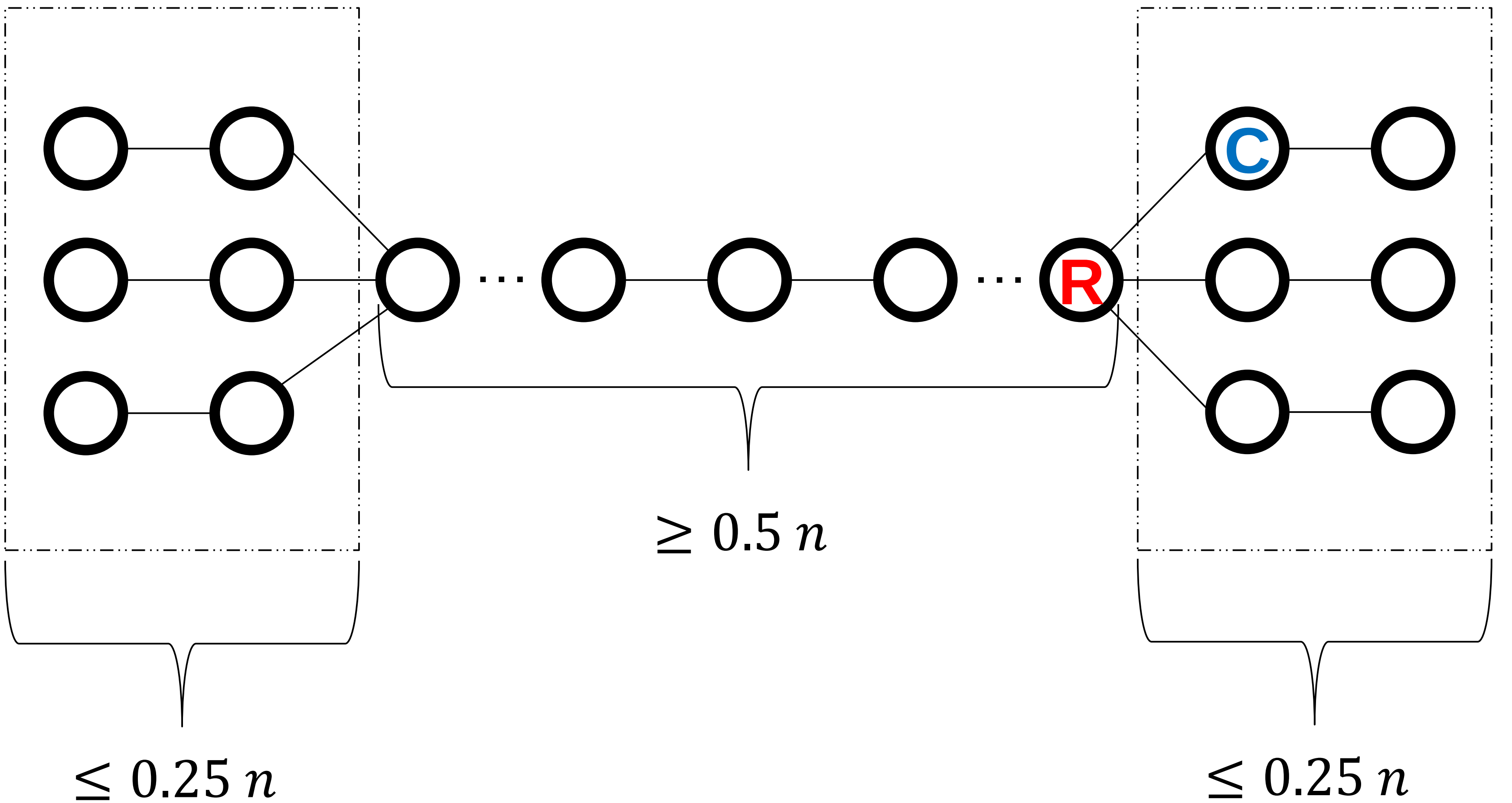

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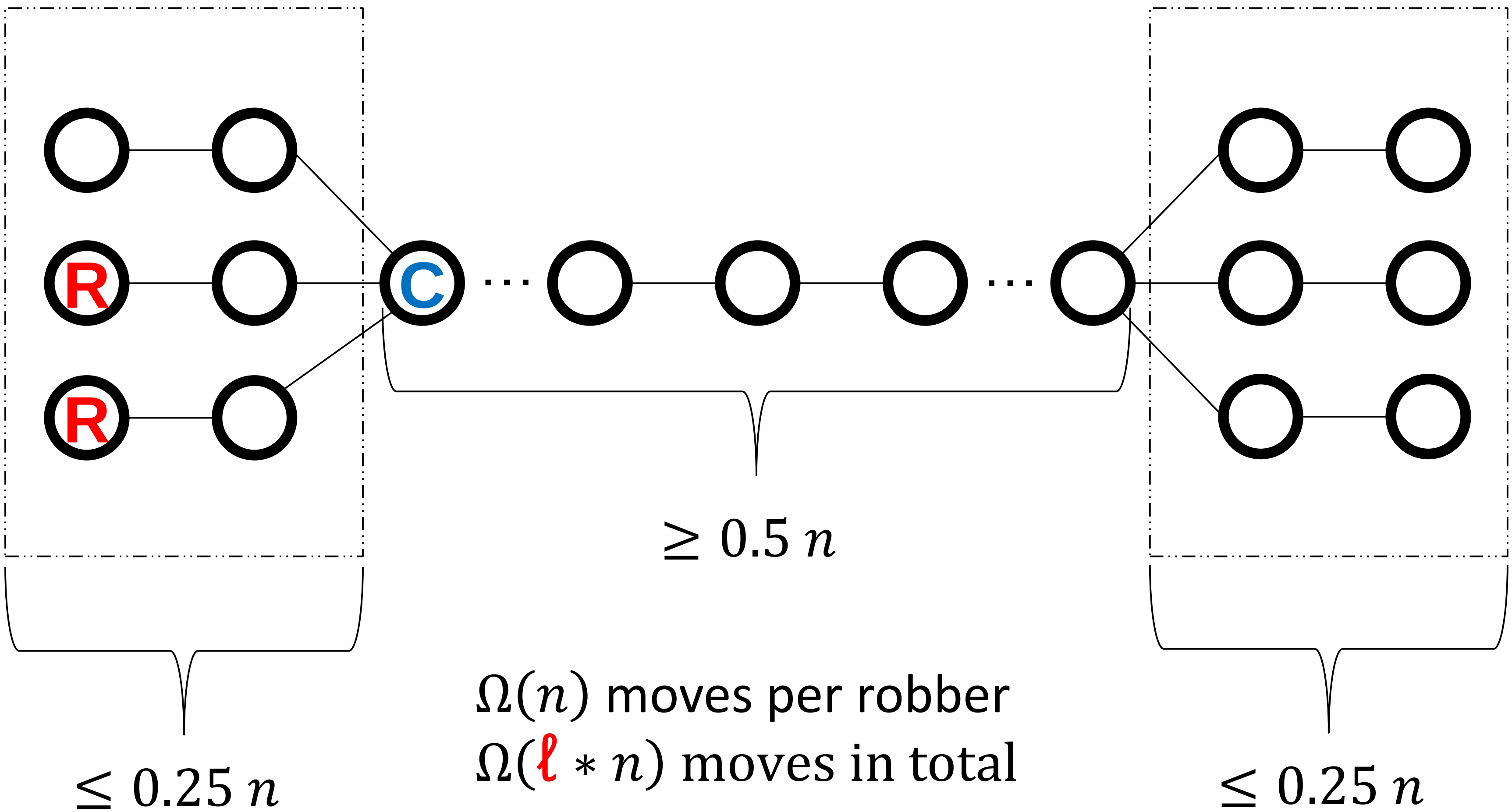
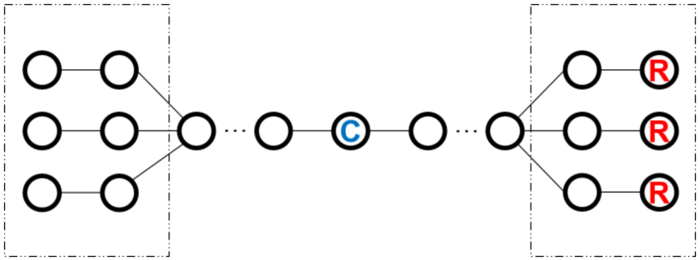


$\Omega(n)$ moves per robber
$\Omega(\ell * n)$ moves in total

Summary so far

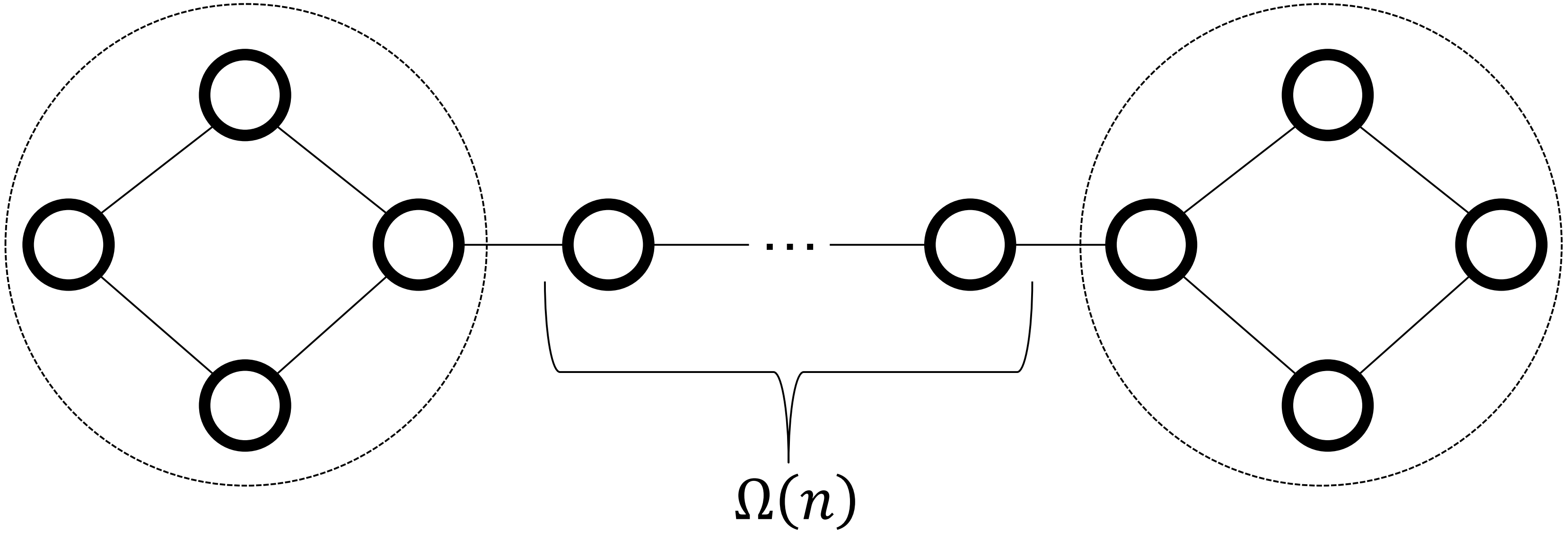
- 1 cop and $\ell \in O(n)$ robbers (in $c(G) = 1$ graphs)
  - $O(\ell * n)$ moves always suffice
  - $\Omega(\ell * n)$ needed in some graphs



What about multiple cops and one robber?

- k cops and 1 robber (in $c(G) = k$ graphs)
  - Best known upper bound: n^{k+1} (Berarducci and Intrigila, 1993)
  - Lower bound?

Let's start with two cops and one robber



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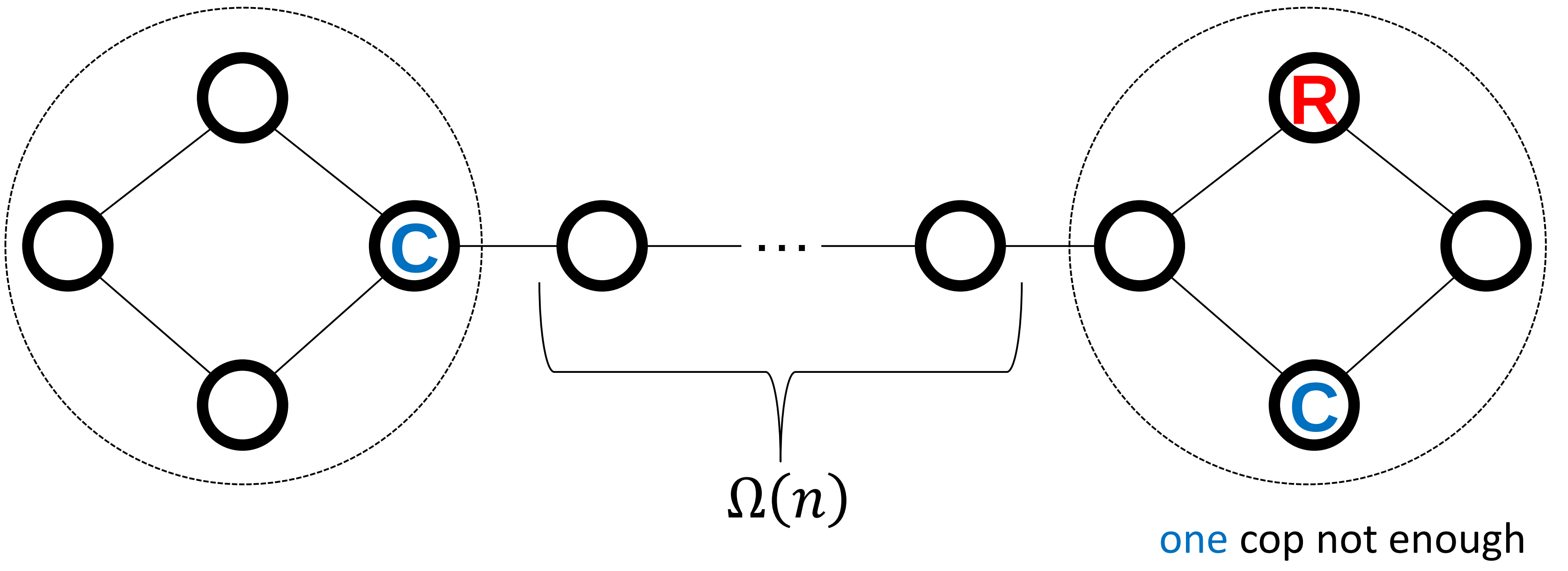

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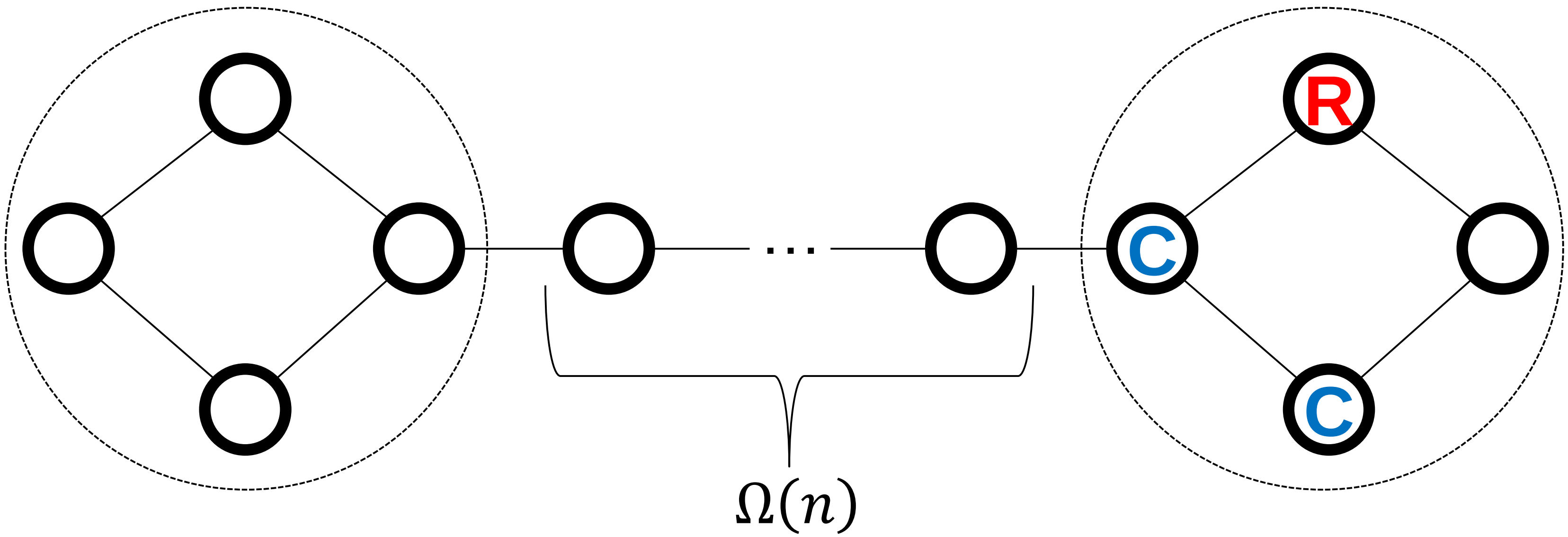
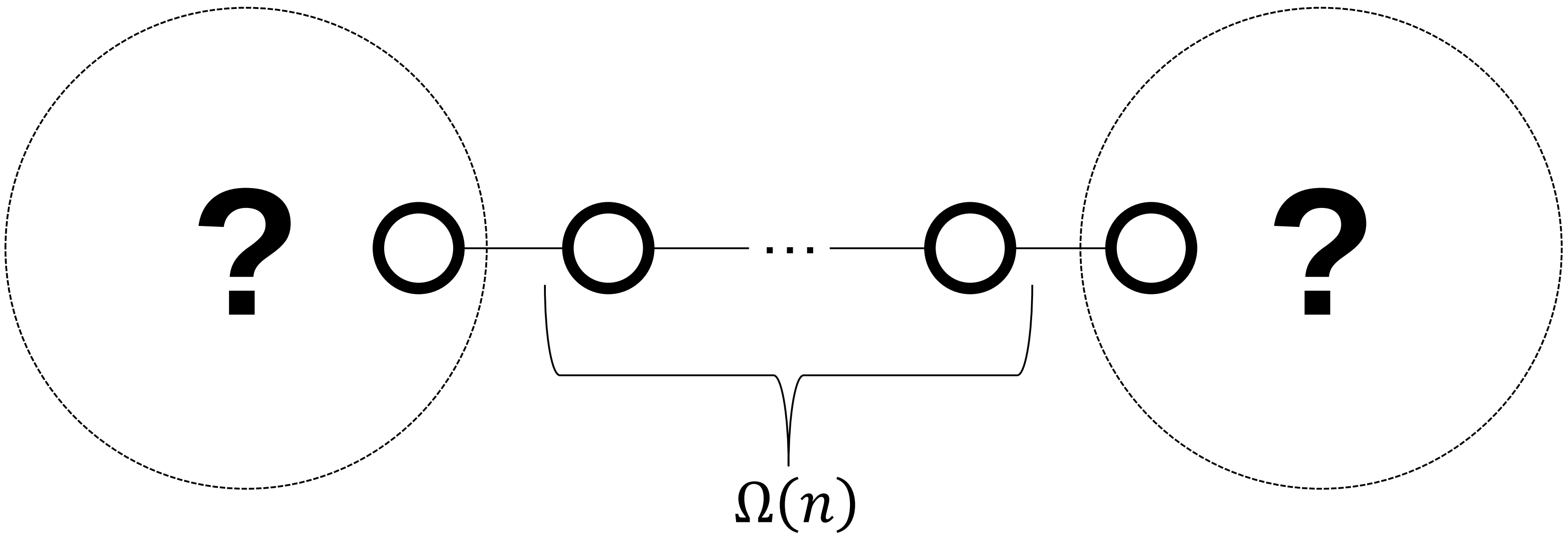


$\Omega(n)$ moves are needed

Beyond two cops?



How large can $c(G)$ be compared to n?

Beyond two cops?

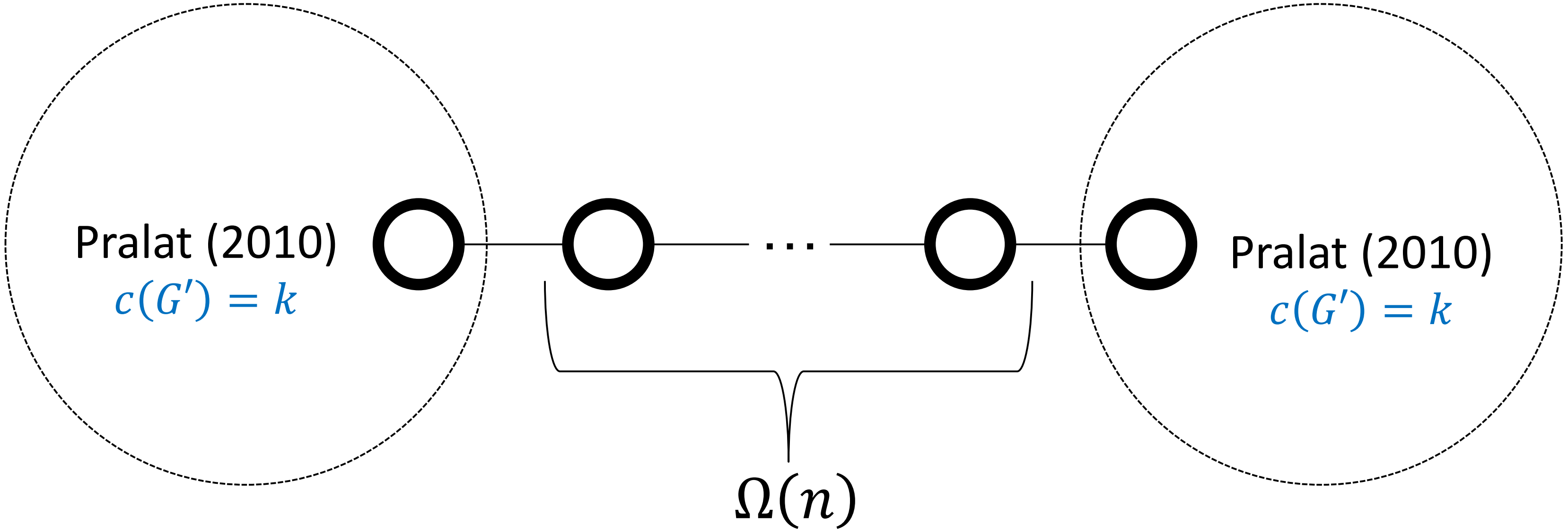
- Aigner and Fromme 1984: 3 for planar graphs
- Meyniel's **conjecture** (1985): $\forall G$: $c(G) \in O(\sqrt{n})$
- Known upper bound: $O\left(\frac{n}{\log n}\right)$ (Chiniforooshan 2008)
- Improved to $O\left(n/\left(2^{(1-o(1))\sqrt{\log n}}\right)\right)$

  (Frieze, Krivelevich, and Loh 2012; Lu and Peng 2012; Scott and Sudakov 2011)

- Pralat (2010): $\exists G'$: $c(G') \in \Omega(\sqrt{n})$

Multiple cops and one robber

Note that $c(G') = k + 1$ may hold!



Robber chooses side with less than $0.5 * c(G)$ cops

Construction has $n \in O(k^2)$ nodes

$\Omega(n)$ moves are needed

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  - $O(\ell * n)$ moves always suffice
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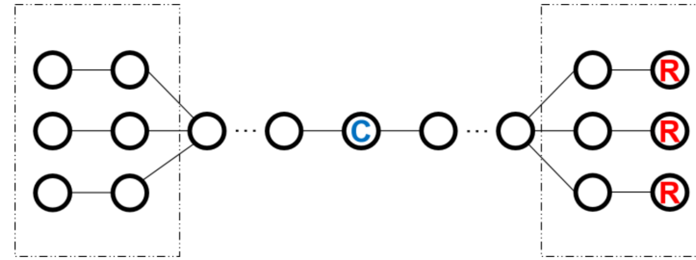
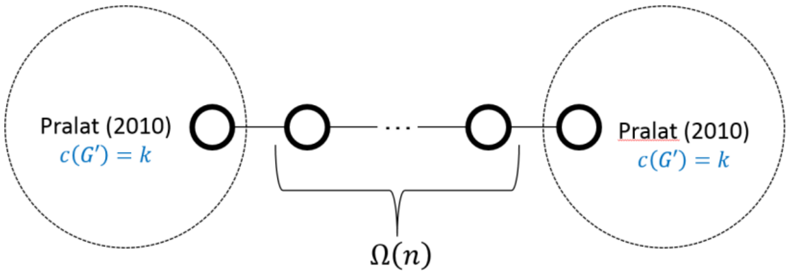


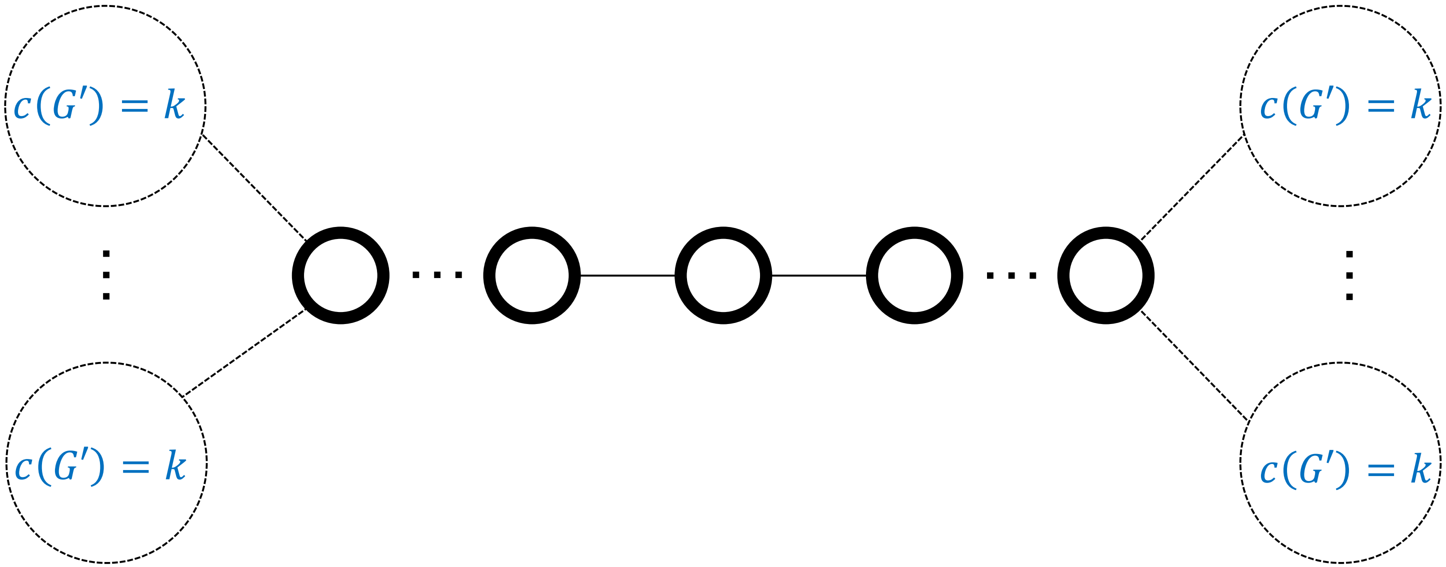
- $k \epsilon O(\sqrt{n})$ cops and 1 robber (in $c(G) = k$ graphs)
  - Best known upper bound: n^{k+1}
  - $\Omega(n)$ moves with $n \in O(k^2)$ nodes



What about multiple cops and multiple robbers?

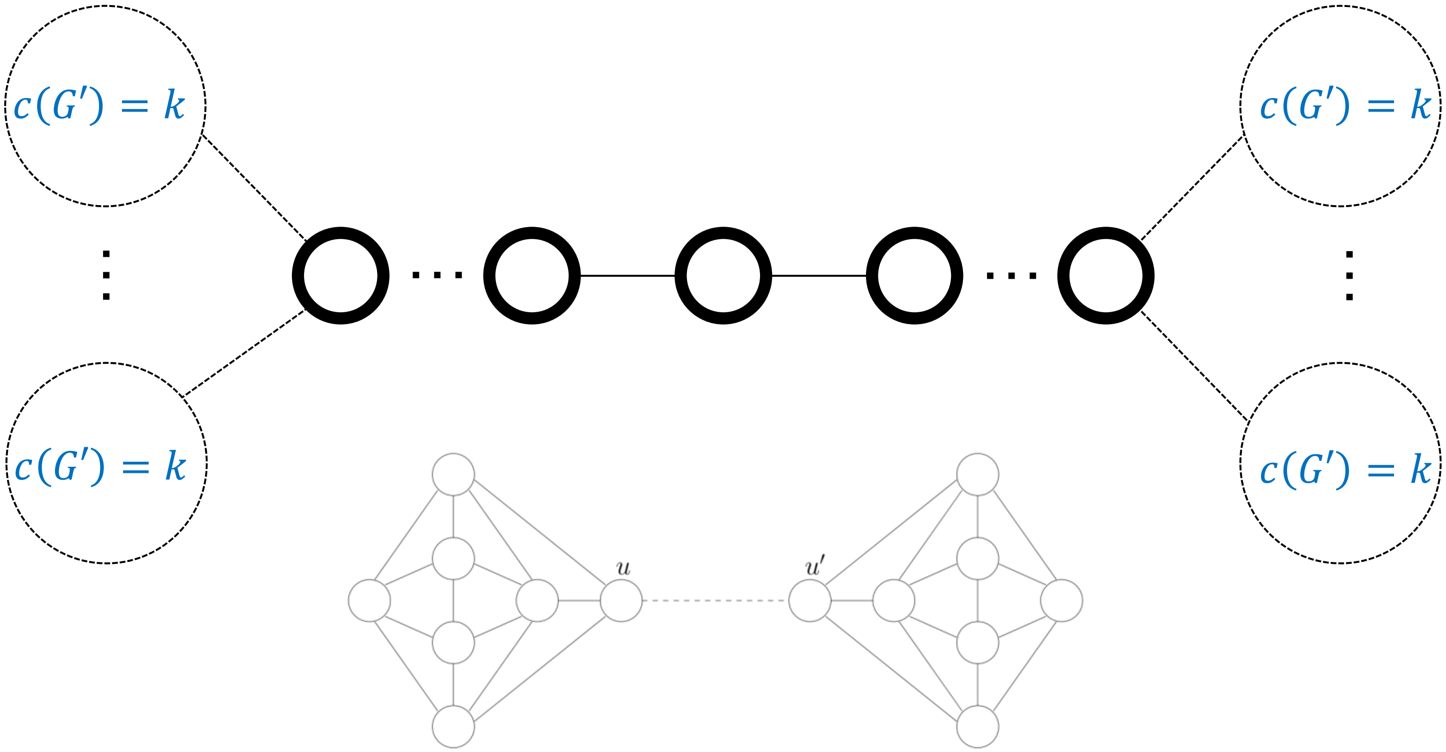
- k cops and ℓ robbers (in $c(G) = k$ graphs)
  - **?**

Multiple cops and multiple robbers



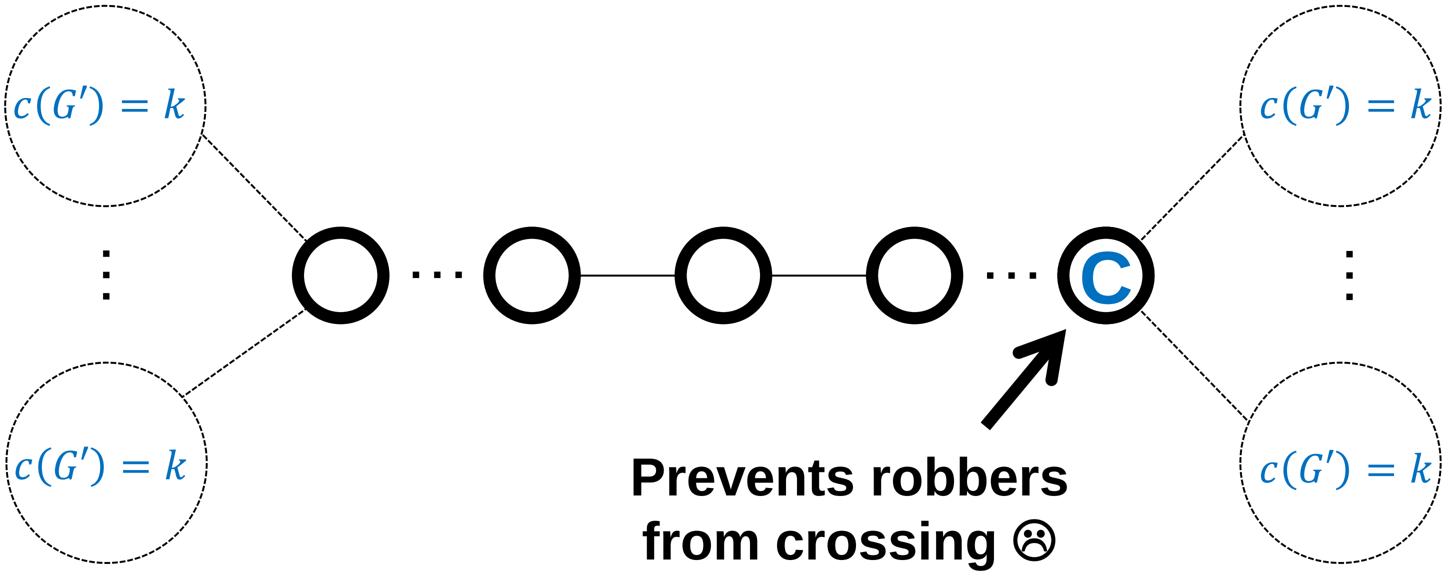
Are we done?

Multiple cops and multiple robbers



Problem: $\boldsymbol{c(G) = k + 1}$ **?**

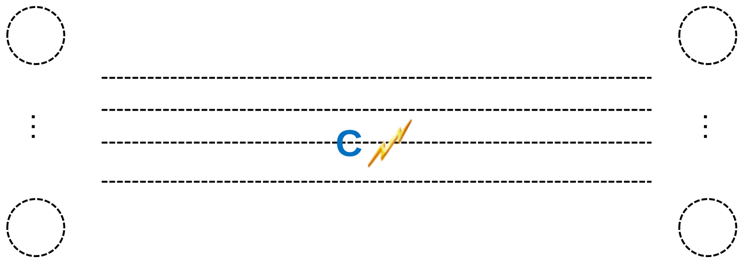
Multiple cops and multiple robbers



Problem: $c(G) = k + 1$?

How to deal with cop $k + 1$?

- Multiple paths do not help much:
  - Cop $k + 1$ „emulates“ robbers
  - Catches fraction each crossing



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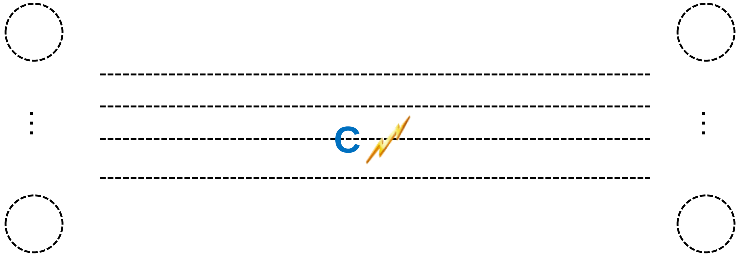
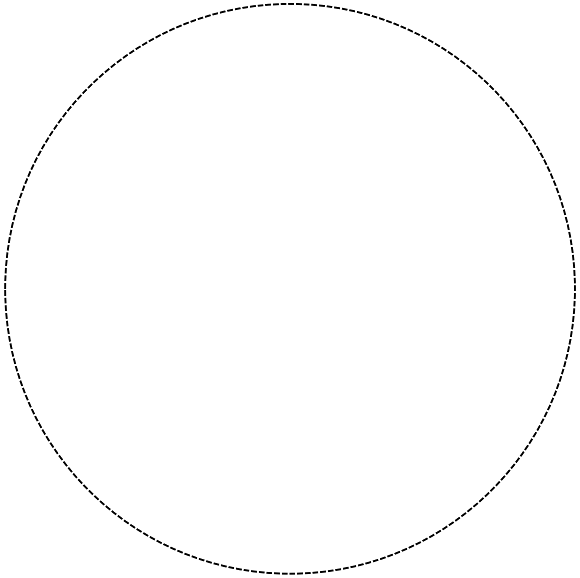


- Better idea:
  - Use a ring



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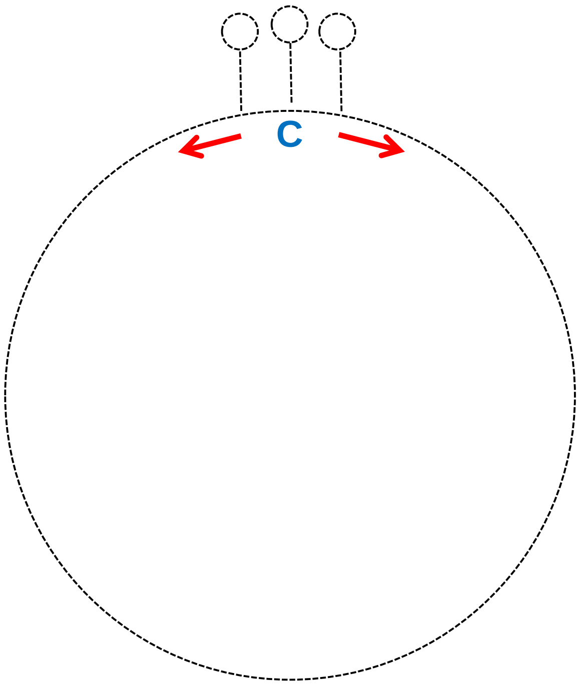

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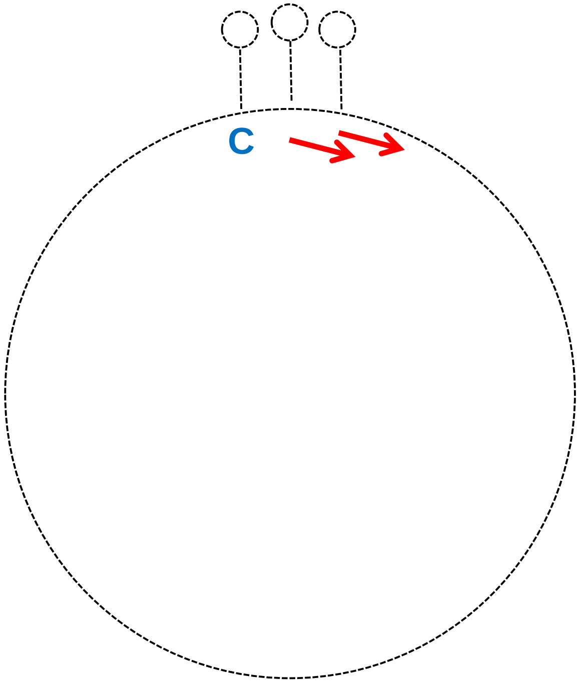

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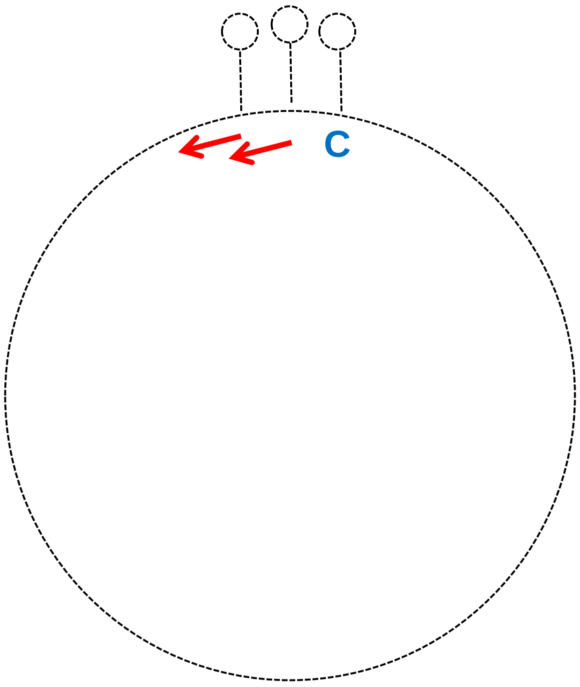
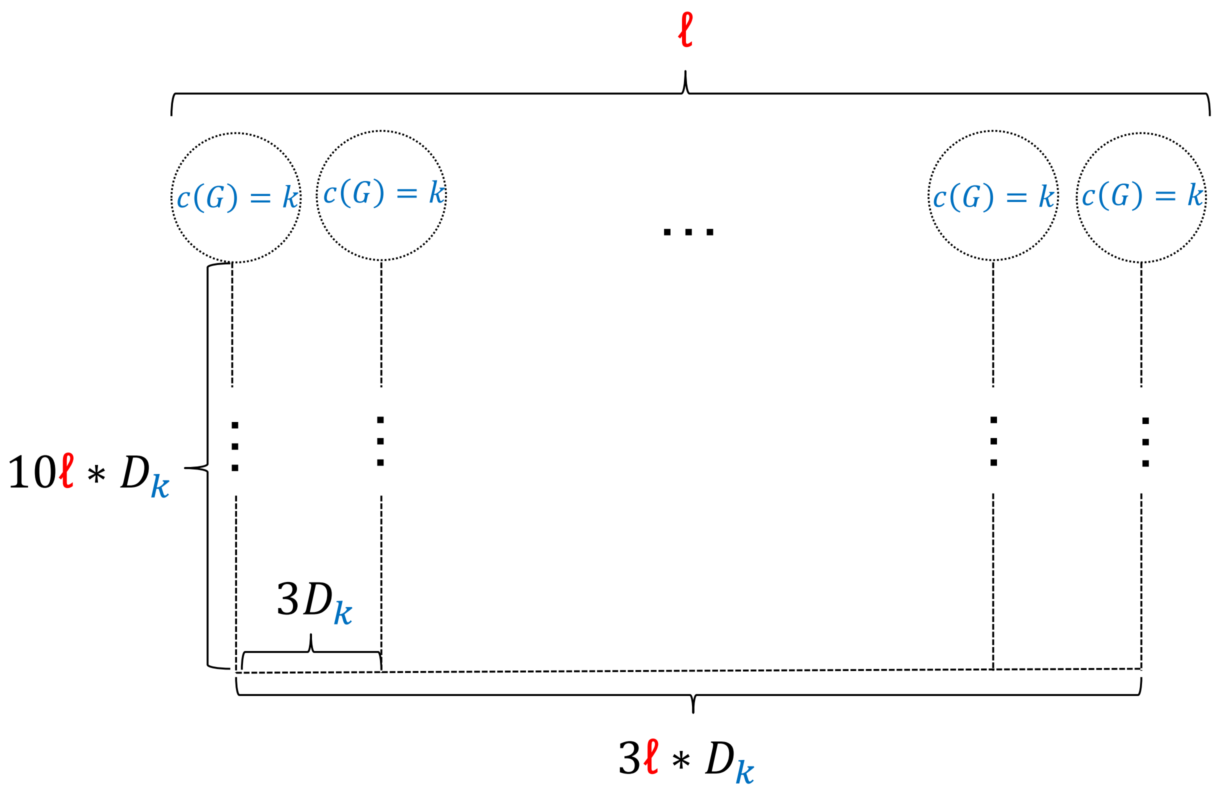
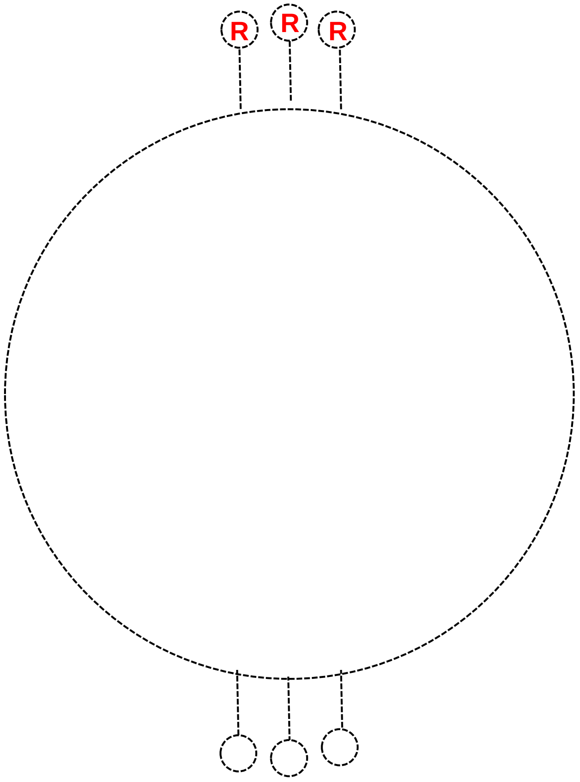


Construction of the ring

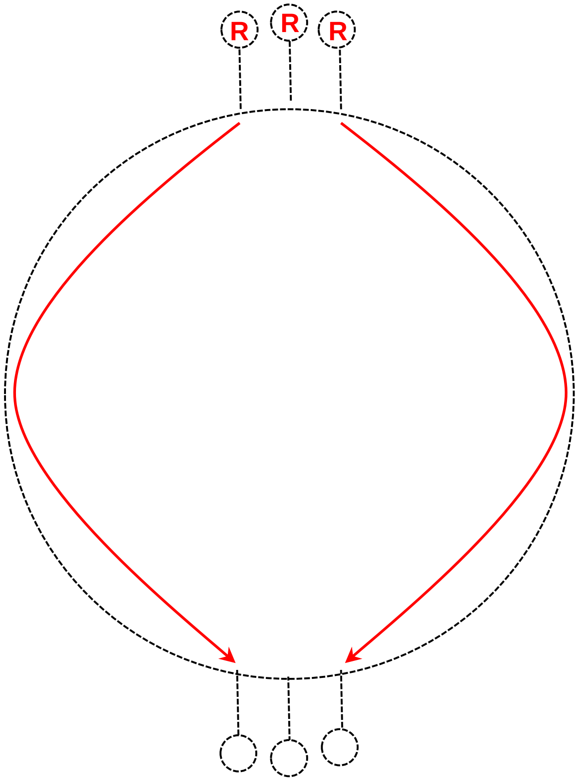


Robber placement



Robbers choose side with less cops

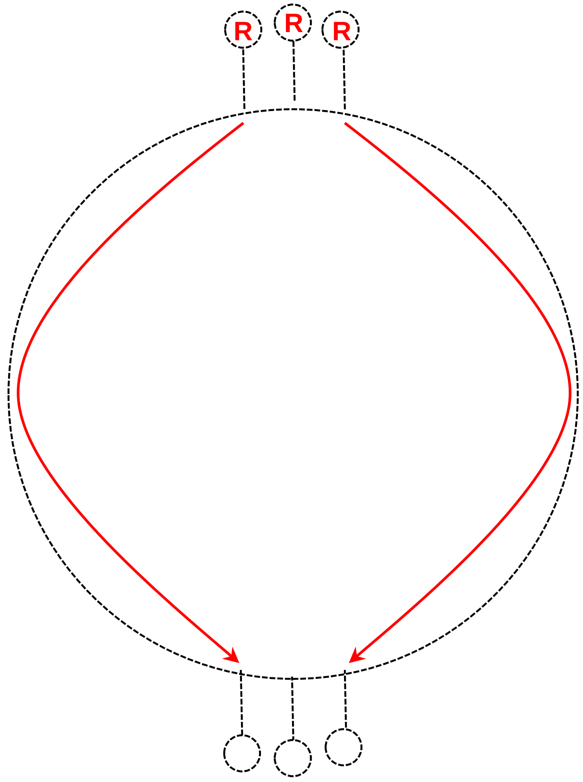
Robber strategy



k cops needed to catch 1 robber in gadget graph

If $c(G) = k$, then all other robbers escape "down"

Robber strategy



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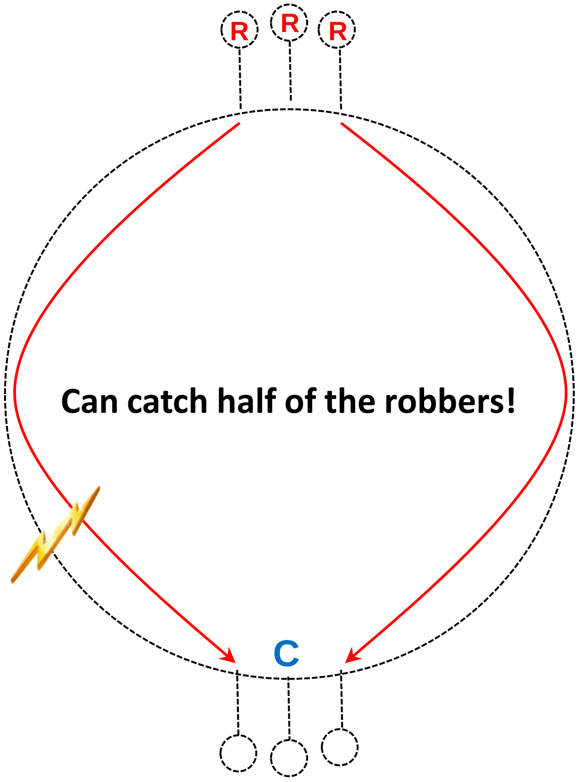
But if $c(G) = k + 1$?

Robber strategy



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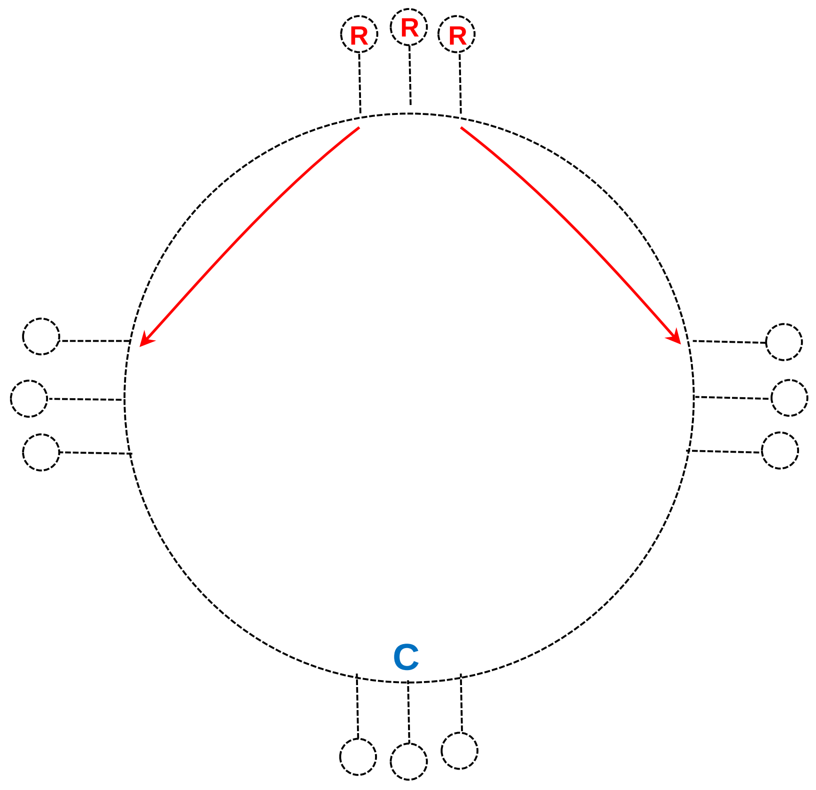

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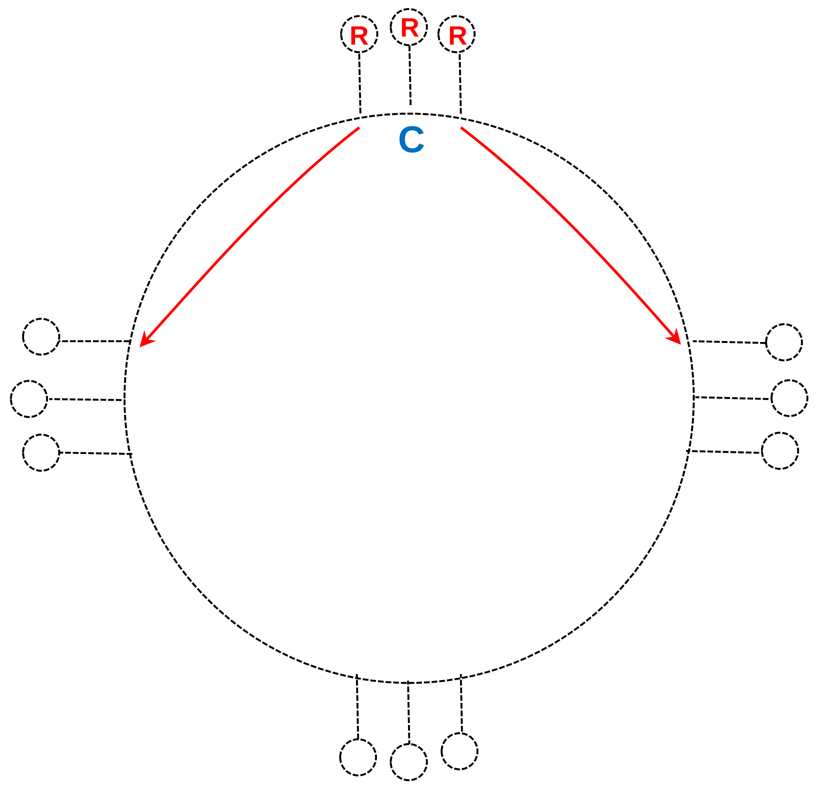

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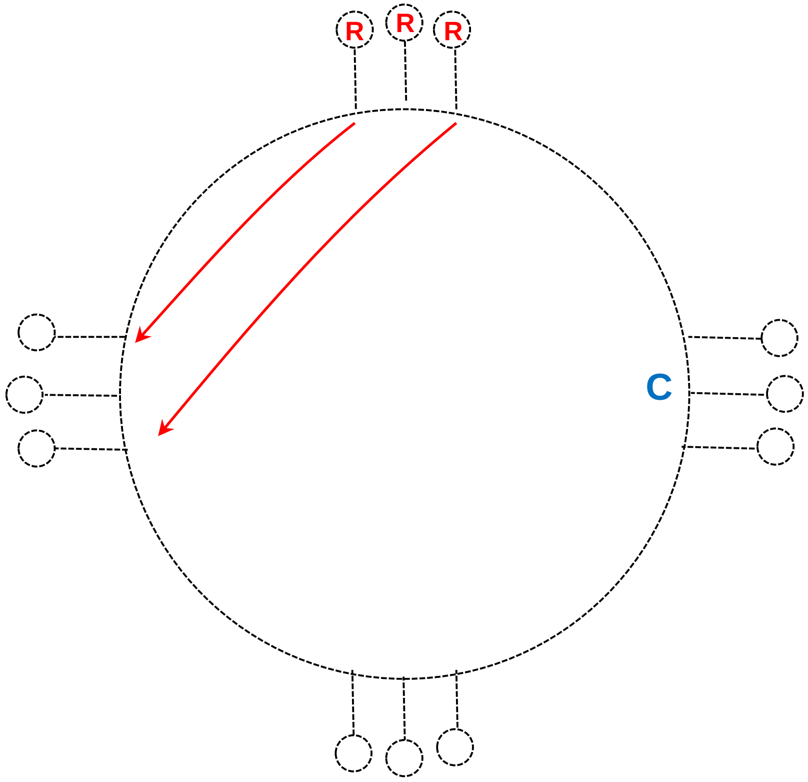

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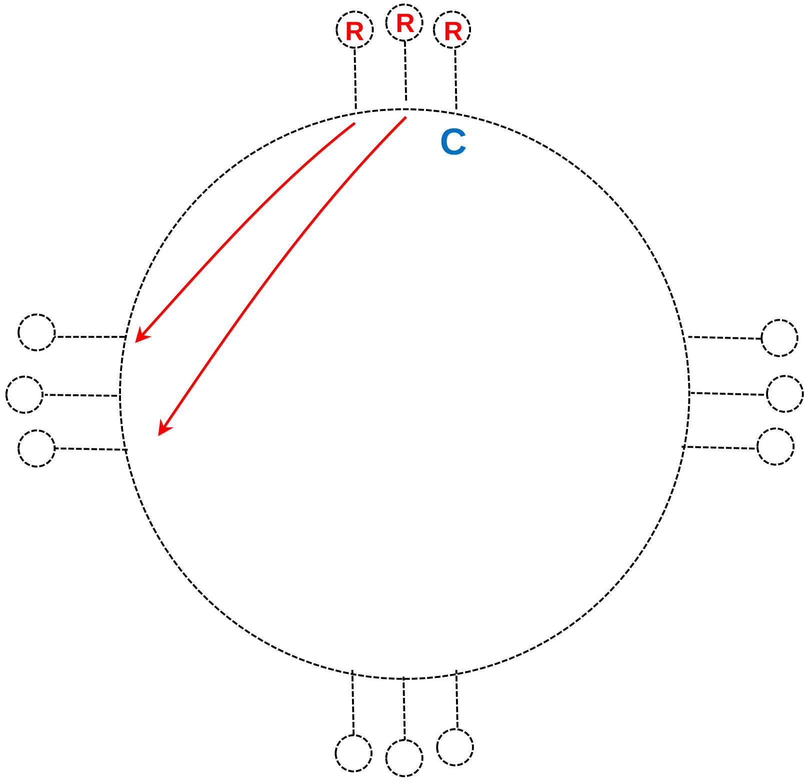
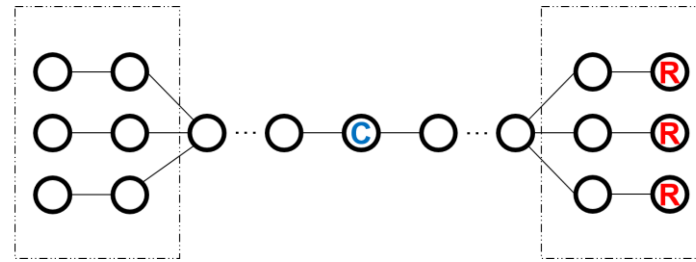


Cops need $\Omega(n)$ moves to catch 2 robbers

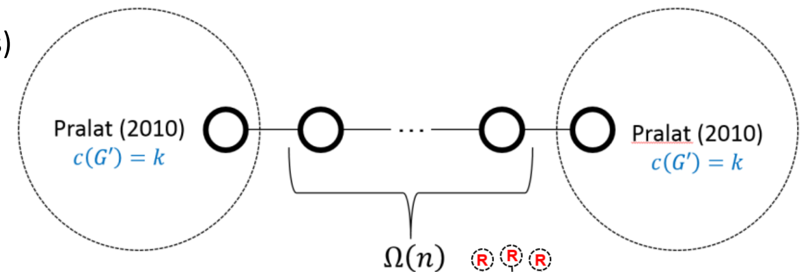
$\Omega(\ell * n)$ moves to catch all robbers

Summary

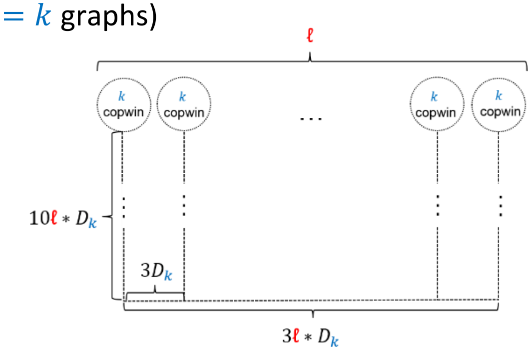
- 1 cop and $\ell \in O(n)$ robbers (in $c(G) = 1$ graphs)
  - $O(\ell * n)$ moves always suffice
  - $\Omega(\ell * n)$ needed in some graphs

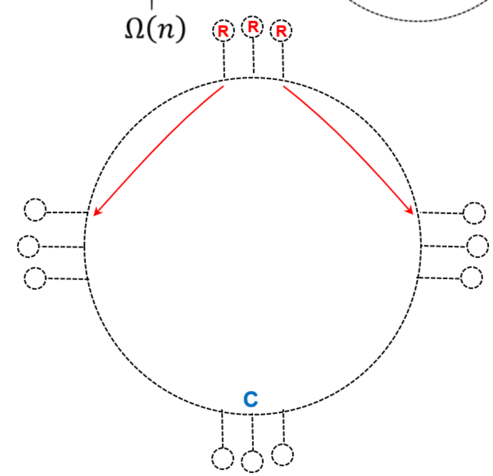


- $k \in O(\sqrt{n})$ cops and 1 robber (in $c(G) = k$ graphs)
  - Best known upper bound: n^{k+1}
  - $\Omega(n)$ moves with $n \in O(k^2)$ nodes



- k cops and ℓ robbers (in $c(G) = k$ graphs)
  - $\Omega(\ell * n)$ moves with
    - $k \in O(\sqrt{n/\ell})$
    - $\ell \in O(\sqrt{n/k})$





- More than n robbers?
  - $\Omega(n^2 \log(\ell/n))$

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