Convergence in (Social) Influence Networks

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Abstract. We study the convergence of influence networks, where each node changes its state according to the majority of its neighbors. Our main result is a new $\Omega(n^2/\log^2 n)$ bound on the convergence time in the synchronous model, solving the classic "Democrats and Republicans" problem. Furthermore, we give a bound of $\Theta(n^2)$ for the sequential model in which the sequence of steps is given by an adversary and a bound of $\Theta(n)$ for the sequential model in which the sequence of steps is given by a benevolent process.

1 Introduction

What do social networks, belief propagation, spring embedders, cellular automata, distributed message passing algorithms, traffic networks, the brain, biological cell systems, or ant colonies have in common? They are all examples of "networks", where the entities of the network are continuously influenced by the states of their respective neighbors. All of these examples of *influence networks* (INs) are known to be difficult to analyze. Some of the applications mentioned are notorious to have long-standing open problems regarding convergence.

In this paper we deal with a generic version of such networks: The network is given by an arbitrary graph $G = (V, E)$, and all nodes of the graph switch simultaneously to the state of the majority of their respective neighbors. We are interested in the stability of such INs with a binary state. Specifically, we would like to determine whether an IN converges to a stable situation or not. We are interested in how to specify such a stable setting, and in the amount of time needed to reach such a stable situation. We study several models how the nodes take turns, synchronous, asynchronous, adversarial, benevolent.

Our main result is for synchronous INs: Each node is assigned an initial state from the set $\{R, B\}$, and in every round, all nodes switch their state to the state of the majority of their neighbors simultaneously. This specific problem is commonly referred to as "Democrats and Republicans", see e.g. Peter Winkler's CACM column [Win08]. It is well known that this problem stabilizes in a peculiar way, namely that each node eventually is in the same state every second round [GO80]. This result can be shown by using a potential bound argument, i.e., until stabilization, in each round at least one more edge becomes "more stable". This directly gives a $\mathcal{O}(n^2)$ upper bound for the convergence time. On the other hand, using a slightly adapted linked list topology, one can see that convergence takes at least $\Omega(n)$ rounds. But what is the correct bound for this classic problem? Most people that worked on this problem seem to believe that the linear lower bound should be tight, at least asymptotically. Surprisingly, in the course of our research, we discovered that this is not true. In this paper we show that the upper bound is in fact tight up to a polylogarithmic factor. Our new lower bound is based on a novel graph family, which has interesting properties by itself. We hope that our new graph family might be instrumental to research concerning other types of INs, and may prove useful in obtaining a deeper understanding of some of the applications mentioned above.

We complement our main result with a series of smaller results. In particular, we look at asynchronous networks where nodes update their states sequentially. We show that in such a sequential setting, convergence may take $\Theta(n^2)$ time if given an adversarial sequence of steps, and $\Theta(n)$ if given a benevolent sequence of steps.

2 Related Work

Influence networks have become a central field of study in many sciences. In biology, to give three examples from different areas, [RT98] study networks in the context of brain science, [AAB$^+$11] study cellular systems and their relation to distributed algorithms, and [AG92] study networks in the context of ant colonies. In optimization theory, believe propagation [Pea82, BTZ$^+$09] has become a popular tool to analyze large systems, such as Bayesian networks and Markov random fields. Nodes are continuously being influenced by their neighbors; repeated simulation (hopefully) quickly converges to the correct solution. Belief propagation is commonly used in artificial intelligence and information theory and has demonstrated empirical success in numerous applications such as coding theory. A prominent example in this context are the algorithms that classify the importance of web pages [BP98, Kle99]. In physics and mechanical engineering, force-based mechanical systems have been studied. A typical model is a graph with springs between pairs of nodes. The entire graph is then simulated, as if it was a physical system, i.e. forces are applied to the nodes, pulling them closer together or pushing them further apart. This process is repeated iteratively until the system (hopefully) comes to a stable equilibrium, [KK89, Koh89, FER91, KW01]. Influence networks are also used in traffic simulation, where nodes (cars) change their position and speed according to their neighboring nodes [NS92]. Traffic networks often use cellular automata as a basic model. A cellular automaton [Neu66, Wol02] is a discrete model studied in many fields, such as computability, complexity, mathematics, physics, and theoretical biology. It consists of a regular grid of cells, each in one of a finite number of states, for instance 0 and 1. Each cell changes its state according to the states of its neighbors. In the popular game of life [Gar70], cells can be either dead or alive, and change their states according to the number of alive neighbors.

Our synchronous model is related to cellular automata, on a general graph; however, nodes change their opinion according to the majority of their neighbors. As majority functions play a central role in neural networks and biological applications this model was already studied during the 1980s. Goles and Olivos [GO80] have shown that a synchronous binary influence network with a generalized threshold function always leads to a fixed point or to a cycle of length 2. This means that after a certain amount of synchronous rounds, each participant has either a fixed opinion or changes its mind in every round. Poljak and Sura [PS83] extended this result to a finite number of opinions. In [GT83], Goles and Tchuente show that an iterative behavior of threshold functions always leads to a fixed point. Sauerwald and Sudholt [SS10] study the evolution of cuts in the binary influence network model. In particular, they investigate how cuts evolve if unsatisfied nodes flip sides probabilistically. To some degree, one may argue that we look at the deterministic case of that problem instead.

In sociology, understanding social influence (e.g. conformity, socialization, peer pressure, obedience, leadership, persuasion, sales, and marketing) has always been a cornerstone of research, e.g. [Kel58]. More recently, with the proliferation of online social networks such as Facebook, the area has become en vogue, e.g. [MMG$^+$07, AG10]. Leskovec et al. [LHK10] for instance verify the balance theory of Heider [Hei46] regarding conformity of opinions; they study how positive (and negative) influence links affect the structure of the network. Closest to our paper is the research dealing with influence, for instance in the form of sales and marketing. For example, [LSK06] investigate a large person-to-person recommendation network, consisting of four million people who made sixteen million recommendations on half a million products, and then analyze cascades in this data set. Cascades can also be studied in a purely theoretical model, based on random graphs with a simple threshold model which is close to our majority function [Wat02]. Rumor spreading has also been studied algorithmically, using the random phone call model, [KSSV00, SS11, DFF11]. Using real data from various sources, [ALP12] show that networks generally have a core of influential (elite) users. In contrast to our model, nodes cannot change their state back and forth, once infected, a node will stay infected. Plenty of work was done focusing on the prediction of influential nodes. One wants to find subset of influential nodes for viral marketing, e.g. [KKT05, CYZ10]. In contrast, [KOW08] studies the case of competitors, which is closer to our model since nodes can have different opinions. However, also in [KOW08] nodes only change their opinion once. However, in all these social networks the underlying graph is fixed and the dynamics of the stabilization process takes place on the changing states of the nodes only. An interesting variant changes the state of the edges instead. A good example for this is matching. A matching is (hopefully) converging to a stable state, based on the preferences of the nodes, e.g. [GS62, KPS10, FKPS10]. Hoefer takes these edge dynamics one step further, as not only the state of the edge changes, but the edge itself [Hoe11].

3 Model Definition

An *influence network* (IN) is modeled as a graph $G = (V, E, o_0)$. The set of nodes V is connected by an arbitrary set of edges E. Each node has an initial opinion (or state) $o_0(v) \in \{R(ed), B(lue)\}$. A node only changes its opinion if a majority of its neighbors has a different opinion. One may consider several options to breaking ties, e.g., using the node's current opinion as a tie-breaker, or weighing the opinions of individual neighbors differently. As it turns out, for many natural tie-breakers, graphs can be reduced to equivalent graphs in which no tie breaker is needed. For instance, using a node's own opinion as a tie-breaker is equivalent to cloning the whole graph, and connecting each node with its clone and the neighbors of its clone.

In this paper we study both synchronous and asynchronous INs. The state of a synchronous IN evolves over a series of rounds. In each round every node changes its state to the state of the majority of its neighbors simultaneously. The opinion of a node v in round t is denoted as $o_t(v)$.

As will be explained in Section 5, the only interesting asynchronous model is the sequential model. In this model, we call the change of opinion of one node a *step*. The opinion of node v after t steps is defined as $o_t(v)$. In general, more than one node may be ready to take a step. Depending on whether we want convergence to be fast or slow, we may choose different nodes to take the next step. If we aim for fast convergence, we call this the *benevolent sequential model*. Slow conversion on the other hand we call the *adversarial sequential model*.

We say that an IN stabilizes if it reaches a state where no node will ever change its opinion again, or if each node changes its opinion in a cyclic pattern with periodicity q. In other words, a state can be stable even though some nodes still change their opinion.

Definition 1. *An IN $G = (V, E, o_0)$ is* stable *at time t with periodicity q, if for all vertices $v \in V : o_{t+q}(v) = o_t(v)$. A* fixed state *of an IN G is a stable state with periodicity* 1. *The* convergence time *c of an IN G is the smallest t for which G is stable.*

Note that since INs are deterministic an IN which has reached a stable state will stay stable.

In this paper we investigate the stability, the convergence time c and the periodicity q of INs in the described models. Clearly, the convergence process depends not only on the graph structure, but also on the initial opinions of the nodes. We investigated graphs and initial opinions that maximize convergence time. In the benevolent sequential in particular, we investigate graphs and sets of initial opinions leading to the worst possible convergence time, given the respectively best sequence of steps.

4 Synchronous IN

A synchronous IN may stabilize in a state where some nodes change their opinion in every round. For example, consider the graph K_2 (two nodes, connected by an edge) where the first node has opinion B and the second node has opinion R. After one round, both vertices have changed their state, which leads to a symmetric situation. This IN remains in this stable state forever with a period of length 2. As has already been shown in [GO80, Win08], a synchronous IN always reaches a stable state with a periodicity of at most 2 after $\mathcal{O}(n^2)$ rounds.

Theorem 1 ([Win08]). *A synchronous IN reaches a stable state after at most $\mathcal{O}(n^2)$ rounds.*

Theorem 2 **([GO80]).** *The periodicity of the stable state in a synchronous IN is at most 2.*

In this paper, we prove this bound to be almost tight.

Theorem 3. *There is a family of synchronous INs with convergence time* $\Omega\left(\frac{n^2}{(\log \log n)^2}\right)$.

Unfortunately, the technical proof of Theorem 3 does not fit into this paper. You can find the full proof in Appendix A. In this section, we instead present a simpler IN with convergence time $\Omega\left(n^{3/2}\right)$.

The basic idea is to construct a mechanism which forces vertices on a simple path graph to change their opinion one after the other. Every time the complete path has changed, the mechanism should force the vertices of the path to change their opinions back again in the same order. To create this mechanism, we introduce an auxiliary structure called transistor, which is depicted in Figure 1.

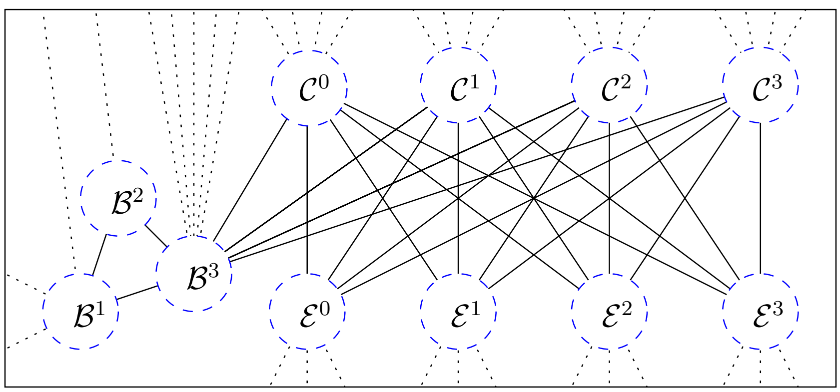

Fig. 1: A transistor $T(4)$. The dotted lines indicate how the transistor will be connected.

Definition 2. *A* transistor *of size* k*, denoted as* $T(k)$*, is an undirected graph consisting of* k *collector vertices* $\mathcal{C} = \{\mathcal{C}^i \mid 0 \leq i \leq k-1\}$*,* k *emitter vertices* $\mathcal{E} = \{\mathcal{E}^i \mid 0 \leq i \leq k-1\}$ *and three base vertices* $\mathcal{B} = \{\mathcal{B}^1, \mathcal{B}^2, \mathcal{B}^3\}$*. All edges between collector and emitter vertices, all edges between any two base vertices, and all edges between collector vertices and the third base vertex exist. Formally:*

$$
\begin{aligned}
T(s) =& (V, E) \\
V =& \mathcal{C} \cup \mathcal{E} \cup \mathcal{B} \\
E =& \{\{u, v\} \mid u \in \mathcal{C}, v \in \mathcal{E}\} \cup \{\{u, \mathcal{B}^3\} \mid u \in \mathcal{C}\} \cup \\
& \{\{u, v\} \mid u, v \in \mathcal{B}, u \neq v\}
\end{aligned}
$$

All nodes in a transistor are initialized with the same opinion $X \in \{R = 1, B = -1\}$. The $3 + k + k^2$ collector edges (dotted edges pointing to the top of Figure 1, including those originating from $\mathcal{B}^1$,$\mathcal{B}^2$ and $\mathcal{B}^3$) are connected to vertices with the constant opinion $-X$, while up to $k^2 - k$ emitter edges (dotted edges pointing to the bottom) and the 2 base edges (dotted edges pointing to the left) may be connected to any vertex. As soon as both base edges advertise opinion $-X$, the transistor will flip to opinion $-X$ in 4 rounds regardless of what is advertised over the emitter edges, i.e., the following sets of vertices will all change their opinion to $-X$ in the given order: $\{\mathcal{B}^1\}$, $\{\mathcal{B}^2, \mathcal{B}^3\}$, $\mathcal{C}$, $\mathcal{E}$.

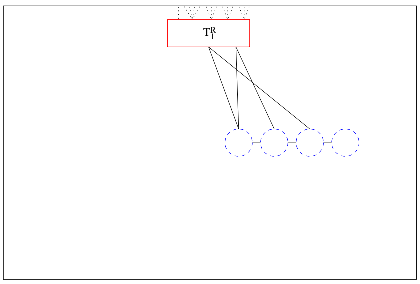

Fig. 2: Path with 4 vertices connected to one transistor $T(3)$.

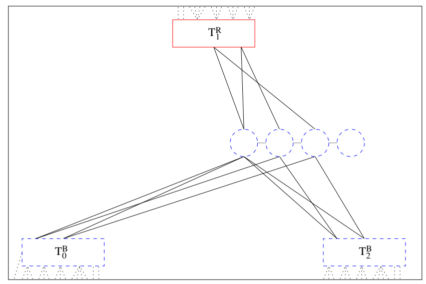

Fig. 3: Path with 4 vertices connected to 3 transistors $T(3)$. Note that transistors at bottom of figures are always upside down.

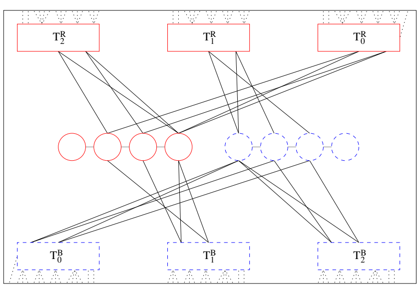

Fig. 4: Two copies of Figure 3 with inverse opinions

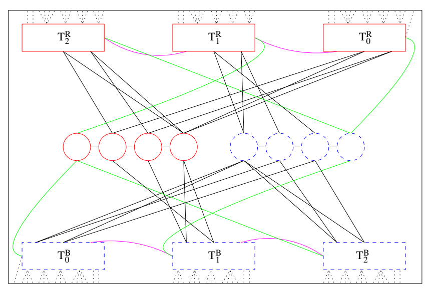

Fig. 5: In this graph, every time the path has run through completely the next transistor will flip, causing the path to run again.

Note that $T(k)$ contains only $\mathcal{O}(k)$ many vertices, yet its emitter vertices can potentially be connected to $\Omega(k^2)$ other vertices. Given a path graph of length $\mathcal{O}(k^2)$ and a transistor $T(k)$, the emitter vertices of the transistor are connected to the path in the following way: The first vertex in the path is connected to exactly two emitter vertices, the last is connected to none and each of the remaining nodes of the path is connected to exactly one emitter vertex. Furthermore, the collector edges of transistors of opinion

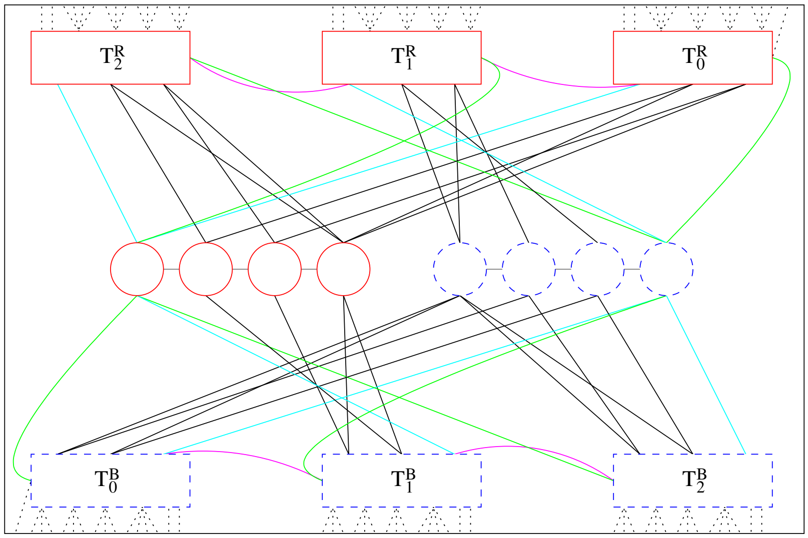

Fig. 6: Final graph in which the paths run 3 times.

X are always connected to constant reservoirs of opinion $-X$. Such a reservoir can be implemented as a clique. An illustration of this graph with $k = 3$ is given in Figure 2. Without loss of generality, we set the initial state of the nodes of the path to B, and that of the transistor to R. As long as the transistor remains red, the path will turn red one vertex at a time. As soon as the transistor flips its opinion to blue (as a result of both base edges having advertised blue) the path will turn blue again, one vertex at a time. To force the path to change k times, k transistors are needed. Each of these transistors (note that we make use of red as well as blue transistors) is connected with the path in the same way as the first transistor. The resulting graph is given in Figure 3. A series of k switches of the complete path can now be provoked by switching transistors of alternating opinions in turns. For the example depicted in the Figures, the switching order of the transistors is given by their respective indices.

Now, a way is needed to flip the next transistor every time the last vertex of the path has changed its opinion. Assume the last vertex has changed to red. It is necessary to flip a red transistor to blue in order to change the path to blue; however, the path changing to red can only cause a blue transistor to turn red. To this end, the graph is extended by a copy of itself with all opinions inverted. The resulting graph is given in Figure 4. As in every round each vertex in the copy is of the opposite opinion than its original, the copy of the last vertex in the path enables us to flip a red transistor to blue as desired. The edges necessary to achieve this (highlighted in green in Figure 5) connect the end of a path to $\mathcal{B}^1$ of each transistor in the other half of the graph. To ensure that the transistors flip in the required order, additional edges (highlighted in magenta in Figure 5) are introduced, connecting an emitter node of each transistor T_i^X to the node $\mathcal{B}^1$ of transistor T_{i+1}^X.

The green edges cause an unwanted influence on the last vertex of the paths. This influence can be negated by introducing additional edges (highlighted in cyan in Figure 6). These edges connect the last vertex of each path with an emitter vertex of each transistor not yet connected to that vertex.

The resulting graph contains $\mathcal{O}(k^2)$ vertices, yet has a convergence time of $\Omega(k^3)$. In terms of the number of vertices n, the convergence time is $n^{3/2}$. The detailed proof in Appendix A shows that this technique can be applied to run the entire graph repeatedly, just as the graph in this section runs two paths repeatedly. This leads to a convergence time of $\Omega(n^{7/4})$. In this new graph, the transistors change back and fourth repeatedly, always taking on the opinion advertised over the collector edges, just like real transistors. When applied recursively $\log \log n$ times, an asymptotic convergence time of $\Omega(n^2/(\log \log n)^2)$ is reached. Since the full proof is long and involved, to complement our formal proof, we also simulated this recursively constructed networks for path lengths of up to 100. Table 1 and Figure 7 show the outcome of this simulation.

path length	#nodes	convergence time
1	10	1
2	12	2
3	96	22
10	494	310
20	1614	3331
30	2010	5701
100	5518	45985

Table 1: Table summarizing the simulated results.

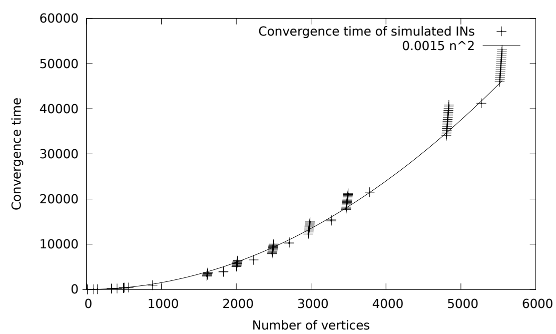

Fig. 7: Shows how our simulation results compare to a quadratic curve. The point clusters arise when for several consecutive path lengths no new transistor is created. Small jumps in the number of vertices indicate that a new transistor was added; big jumps indicate that a new layer of transistors was added.

5 Sequential IN

To complement our results for the synchronous model, we consider an asynchronous setting in this section. In an asynchronous setting, nodes can take steps independently of each other, i.e. subsets of nodes may reassess and change their opinion concurrently. Unfortunately, in such a setting, convergence time is not well defined. To see this, consider a star-graph where the center has a different initial opinion than the leaves. An adversary may arbitrarily often chooses the set of all nodes to reassess their opinion. After r such rounds the adversary chooses only the center node. Now this IN stabilizes, after r rounds for an arbitrary $r \to \infty$. In other words, asynchrony in its most general form is not well defined, and we restrict ourselves to sequential steps only, whereas a step is a single node changing its opinion. The sequence of steps is chosen by an adversary which tries to maximize the convergence time. Note that the convergence upper bound presented in Lemma 1 implies immediately that the IN stabilizes in a fixed state.

Lemma 1. *A sequential IN reaches a fixed state after at most $\mathcal{O}(n^2)$ steps.*

Proof. Divide the nodes into the following two sets according to their current opinion: $S_R = \{v \mid o(v) = R\}$ and $S_B : \{v \mid o(v) = B\}$. If a node changes its opinion, it has more neighbors in the opposite set than in its current set. Therefore the number of edges $X = \{\{u, v\} \mid u \in S_R, v \in S_B\}$ between nodes in set S_R and set S_B is strictly decreasing. Each change of opinion reduces the number of edges of X by at least one. Therefore the number of steps is bounded by the number of edges in X. In a graph G with n nodes $|X|$ is at most $n^2/4$, therefore at most $\mathcal{O}(n^2)$ steps can take place until the IN reaches a fixed state. □

It is more challenging to show that this simple upper bound is tight. We show a graph and a sequence of steps in which way an adversary can provoke $\Omega(n^2)$ convergence time.

Lemma 2. *There is a family of INs with n vertices such that a fixed state is reached after $\Omega(n^2)$ steps.*

Algorithm 1 Adversarial Sequence

```
S ← ()
for i = 0 to n/3 do
    S = reverse(S);
    S ← (i, S);
    for all x ∈ S do
        take step x;
    end for
end for
```

Proof. Consider the following graph G with n nodes. The nodes are numbered from 0 to $n-1$, whereas nodes with an even id are initially assigned opinion B and nodes with an odd id are assigned opinion R. See also Figure 8. All even nodes with $id \leq n/3$ are connected to all odd nodes. All odd nodes with $id \leq n/3$ are connected to all even nodes respectively. In addition an even node with $id \leq n/3$ is connected to nodes $\{0, 2, 4, \ldots, n - 2 \cdot id - 2\}$, respectively an odd node with $id \leq n/3$ is connected to nodes $\{1, 3, 5, \ldots, n - 2 \cdot id - 3\}$. For example, node 0 is a neighbor of all nodes, whereas node 1 is neighbor of all nodes except the nodes $n - 1$ and $n - 3$. Note that

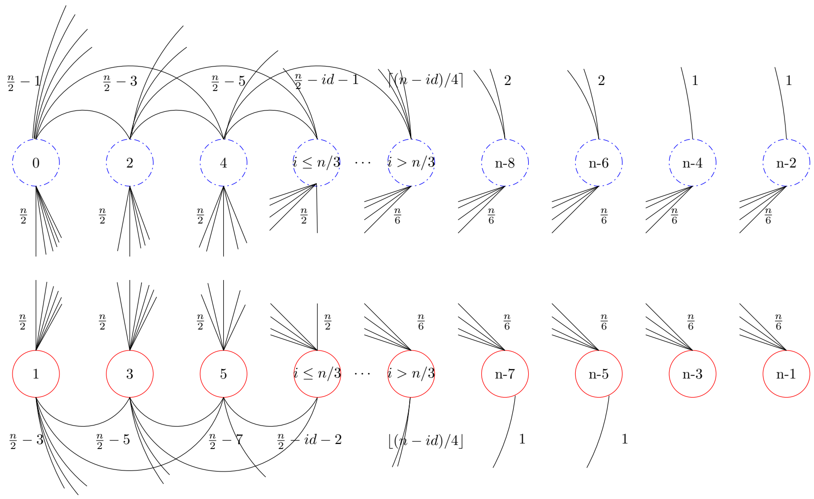

Fig. 8: In this graph an adversary can provoke $\Omega(n^2)$ changes of opinion.

each node i with $i \leq n/3$ is connected to all other nodes with $id \leq n/3$. For each node v the change potential $P(v)$ is defined as:

$$P(v) = |\{u \mid o(u) \neq o(v)\}| - |\{u \mid o(u) = o(v)\}|$$

Put differently, if the change potential of a node is larger than 0, and it is requested to reassess its opinion, it takes a step. A large change potential of a node v, means that many neighbors of v have the opposite opinion from v. If a neighbor of v with the same opinion takes a step, $v's$ change potential $P(v)$ is increased by 2. On the other hand, if a neighbor changes from the opposite opinion to the same opinion as node v, $P(v)$ is decreased by 2. If v itself changes its opinion, its change potential turns from p to $-p$. The change potential of v is basically the number of edges by which the total number of edges between set S_B and set S_R is reduced if v changes its opinion. As the total amount of steps is bounded by the number of edges between set S_B and S_R, a node v with $P(v) = p$ reduces the remaining number of possible changes by p if it takes a step. E.g. in the previously constructed graph G, the first nodes have the following change potential: $P(0) = 1, P(1) = 3, P(2) = 3, P(3) = 5$ Generally, node i has a change potential $P(i) = n/2 - (n/2 - i - 1) = i + 1$ if i is even respectively $P(i) = i + 2$ if i is odd. In order to provoke as many steps as possible, the adversary selects the nodes which have to reassess their opinion according to the following rule: He chooses the node with the smallest id for which $P(v) = 1$. Therefore each step reduces the remaining number of possible steps by 1. G is constructed in such a way, that a step from a node triggers a cascade of steps from nodes which have already changed their opinion whereas each change reduces the overall potential by 1.

The adversary chooses the nodes in phases according to algorithm 1. Phase i starts with the selection of node i followed by the selections of all nodes with $id < i$, where the adversary chooses the nodes in the reverse order than it did in round $i-1$. Phase 0 consists of node 0 changing its opinion, in phase 1 node 1 and then node 0 make steps, and in phase 2 the nodes change in the sequence $2, 0, 1$. As a node v can only change its opinion if $P(v) > 0$, we need to show that this is the case for each node v which is selected by the adversary. It is sufficient to show that each node which is selected has a change potential of 1.

We postulate:

(i) At the beginning of phase i, the following holds: $P(i) = 1$ and $\forall v < i : o(v) = o(i)$.

(ii) Each node the adversary selects has change potential 1 and each node with $id \leq i$ is selected eventually in phase i.

(iii) At the end of phase i, all nodes with $id \leq i$ have opinion R if i is even and opinion B if i is odd.

We prove (i), (ii) and (iii) by induction. Initially, part (i) holds, as no node with $id < 0$ exists and as node 0 is connected to $n/2$ nodes with opinion R and to $n/2 - 1$ nodes with opinion B and therefore has change potential 1. In phase 0 only node 0 is selected, therefore part (ii) of holds as well. Node 0 changed its opinion and has therefore at the end of phase 0 opinion R, therefore part (iii) holds as well.

Now the induction step: To simplify the proof of part (i) of we consider odd and even phases separately. Consider an odd phase i. At the start of phase i, no node with $id \geq i$ has changed its opinion yet. Therefore node i still has its initial opinion $o(i) = R$. According to (iii), each node with $id \leq i-1$ has at the end of phase $i-1$ opinion $R = o(i)$. So $(i+1)/2$ neighbors of i have compared to the initial state, changed their opinion from B to R. If a neighbor u of a node v with a different opinion than v changes it, $v's$ change potential is decreased by 2. Therefore node $i's$ initial change potential $P_{t_0}(i) = n/2 - (n/2 - i - 2) = i + 2$ is decreased by $2 \cdot (i+1)/2 = i+1$ and is therefore $P(i) = i + 2 - (i+1) = 1$ at the beginning of phase i. Therefore (i) holds before an odd phase.

Now consider an even phase i. At its start, all nodes with $id \geq i$ still have their initial opinion. Therefore node i has opinion $o(i) = B$. According to (iii) each node with $id \leq i-1$ has at the end of phase $i-1$ opinion $B = o(i)$. As node $i's$ initial change potential was $P_{t_0}(i) = n/2 - (n/2 - i - 1) = i + 1$ and $i/2$ neighbors of i changed from opinion R to opinion B compared to the initial state, $i's$ new change potential is calculated as $P(i) = i + 1 - 2 \cdot i/2 = 1$. Therefore (i) holds before an even phase, hence (i) holds.

To prove part (ii) let v be the last node which was selected in phase $i-1$. As v was selected, it had according to (ii) a change potential of 1. If a node changes its opinion, its change potential gets inversed. Therefore node v had at the beginning of phase i a change potential of -1. In addition, node v is by construction a neighbor of node i and has according to (i) at the start of phase i the same opinion as node i. As node i changes its opinion, node $v's$ change potential is increased by 2. Therefore $v's$ new change potential is again $-1 + 2 = 1$, when it is selected by the adversary. The same argument holds for the second last selected node u. After it was selected in phase $i-1$

its change potential was -1. Then v has changed its opinion which led to $P(u) = -3$. As node i and node v changed their opinions in phase i, $P(u)$ was again 1. Hence if the adversary selects the nodes in the inverse sequence as in phase $i-1$, each selected node has a change potential of 1 and is selected eventually. Therefore (ii) holds.

As node i and all nodes with $id \leq i-1$ had at the beginning of phase i the opinion $o(i)$ according to (iii) and all nodes have changed their opinion in phase i according to (ii), all nodes with $id \leq i$ must have the opposite opinion at the end of phase i, namely R if i is even or B otherwise. Therefore (iii) holds as well.

We now have proven that in phase i, i nodes change their opinion. As the adversary starts n/3 phases, the total number of steps is $1/2 \cdot n/3 \cdot (n/3 - 1) \in \Omega(n^2)$. □

Directly from Lemma 1 and Lemma 2, we get the following theorem.

Theorem 4. *A worst case sequential IN reaches a fixed state after $\Theta(n^2)$ steps.*

We have seen, that with an adapted graph and an adversary an IN takes up to $\Theta(n^2)$ steps until it stabilizes. But how bad can it get, if the process is benevolent instead?

Theorem 5. *An IN with a benevolent sequential process reaches a fixed state after $\Theta(n)$ steps.*

Proof. A benevolent process needs $\Omega(n)$ steps to reach a stable state. This can be seen by considering the complete graph K_n with initially $\lfloor n/2 \rfloor - 1$ red nodes and $\lceil n/2 \rceil + 1$ blue nodes. Independently of the chosen sequence this IN needs exactly $\lfloor n/2 \rfloor - 1$ steps to stabilize because the only achievable stable state is all nodes being blue. To proof that the number of steps is bounded by $\mathcal{O}(n)$ we define the following two sets: The set of all red nodes which want to change: $C_R i = \{v \mid o(v) = R \wedge P(v) > 0\}$ and the set of all blue nodes which want to change: $C_B = \{v \mid o(v) = B \wedge P(v) > 0\}$. A benevolent process chooses nodes in two phases. In the first phase it chooses nodes from C_B until the set is empty. During this phase, it may happen that additional nodes join C_B (e.g. a leaf of a node $v \in C_B$, after v made a step). However, no node which left C_B will rejoin, as those nodes turned red and can not turn blue again in this phase. In the second phase, the benevolent process chooses nodes from C_R until this set is empty. The set C_B will stay empty during the second phase since nodes turning blue can only reinforce blue nodes in their opinion. Both phases take at most n steps, therefore proving our upper bound. □

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Appendix

A Synchronous IN

A.1 Transistor

In this section, we formally define how a transistor must be connected to the rest of the graph and how it behaves in that case. The symbol T is used to denote a particular graph as well as instances of that graph, which are induced subgraphs; the symbols $\mathcal{C}(T), \mathcal{E}(T), \mathcal{B}^x(T)$ are used to denote the respective vertex sets of an instance T. In order to talk about how a transistor should fit in a network, we introduce the outside influence function, which specifies how an induced subgraph is influenced by the rest of the graph.

Definition 3. *For any vertex $v \in V^H$, where $H = (V^H, E^H)$ is an induced subgraph of $G = (V, E)$, the outside influence at time t exerted on said vertex is equal to the following: $I_t^H(v) = o_0(v) \cdot \sum_{u \in \{u' | \{v,u'\} \in E \setminus E^H\}} o_t(u)$.*

So if $I_t^H(v)$ is positive, node v will be influenced by the rest of the graph to stick with its initial opinion or change back to it; if $I_t^H(v)$ is negative, v will be influenced to change away from its initial opinion. The upper right indices are sometimes left out if they are clear from the context. We are now able to formally specify an outside influence range in which the transistor will operate correctly.

Definition 4. *An instance $T = (V^T, E^T)$ of a transistor $T(k)$ with $k \geq 1$ and with outside influence $I_t^T(\cdot)$ is* correctly accessed *with initial opinion X if and only if the following conditions hold.*

(i) $o_0(v) = X$ *for all* $v \in V^T$
(ii) $I_t(v) = -k$ *for all t and all* $v \in \mathcal{C}(T)$
(iii) $|I_t(v)| \leq k - 1$ *for all t and all* $v \in \mathcal{E}(T)$
(iv) $I_t(\mathcal{B}^1(T)) \in \{-3, -1, 1\}$ *for all t*
(v) $I_t(\mathcal{B}^2(T)) = -1$ *for all t*
(vi) $I_t(\mathcal{B}^3(T)) = -(k + 1)$ *for all t*

Note that if all collector edges in Figure 1 advertise R then the conditions (ii) through (vi) are fulfilled independent of that the other edges connecting the transistor to the rest of graph advertise. Because of (ii) there is a strong outside influence on the collector vertices to change their opinion. The first round t where $I_t(\mathcal{B}^1(T))$ is equal to -3 (both base edges in Figure 1 advertise R), will cause that eventually all vertices in the transistor flip their opinion to $(-X)$. We call this event t flip time $t^f(T)$ of T.

Lemma 3. *If an instance $T = (V^T, E^T)$ of a transistor $T(k)$ is correctly accessed with initial opinion X, then the following statements hold.*

(i) All vertices $v \in V^T$ are of opinion $o_t(v) = X$ for all $t \leq t^f(T)$

(ii) All vertices $v \in \mathcal{E}(T)$ are of opinion $o_t(v) = X$ for all $t^f(T) < t \leq t^f(T) + 3$
(iii) All vertices $v \in V^T$ are of opinion $o_t(v) = (-X)$ for all $t \geq t^f(T) + 4$

Proof. First note that the vertices in $\mathcal{E}(T)$ can not change their opinion until at least one in $\mathcal{C}(T)$ has done so since they each have an outside influence of at most $(k-1)$ times $(-X)$ after (iii) of definition 4 and an inside influence of k times X from the k vertices in $\mathcal{C}(T)$. Similarly, the vertices in $\mathcal{C}(T)$ can not change their opinion until $\mathcal{B}^3(T)$ and $\mathcal{B}^2(T)$ have done so and they in turn have to wait for $\mathcal{B}^1(T)$ to change. Finally, $\mathcal{B}^1(T)$ will only change after the flip time $t^f(T)$. This takes care of (i) and (ii). To see that (iii) is true note that after the flip event the sets $(\{\mathcal{B}^1(T)\}, \{\mathcal{B}^2(T), \mathcal{B}^3(T)\}, \mathcal{C}(T), \mathcal{E}(T))$ will indeed all change their opinion in the given order and that even if after the flip event the outside influence of $\mathcal{B}^1(T)$ will go back to some number > -3, the process can still not be stopped or reversed. □

A.2 Counters

The final graph for our intended lower bound is a recursively defined counter. In this section, we present the base case in form of a simple 2-Path graph as well as the recursive case. In the latter, a counter is combined with a number of transistors to form a bigger counter as suggested in the proof outline.

A counter $K = (H = (V, E), \mathcal{I}(\cdot), \mathcal{R}^R, \mathcal{R}^B, \mathcal{S}(\cdot))$ consists of a graph $H = (V, E)$, a function $\mathcal{I} : V \to \mathbb{Z}$, specifies the valid range of outside influence, two special interest vertices $\mathcal{R}^R, \mathcal{R}^B \in V$, which indicate when the graph has finished running, and an initial configuration $\mathcal{S}(\cdot)$.

We will postpone the definition of the axioms a counter must satisfy, and first describe how a counter is properly connected and accessed since the behavior of a counter need only be defined if it is connected and accessed correctly.

Definition 5. *A counter $K = (H = (V^H, E^H), \mathcal{I}(\cdot), \mathcal{R}^R, \mathcal{R}^B, \mathcal{S}(\cdot))$ is* correctly accessed *and* correctly initialized *from t_1 to t_2 if and only if the following condition holds.*

(i) For all v in V^H the initial state of v is set by $o_{t_1}(v) = \mathcal{S}(v)$.
(ii) For all X and all $t_1 \leq t \leq t_2$ the outside influence is given by $I_t^H(v) = \mathcal{I}(v)$

A counter is considered *reversely initialized* if (i) is changed to $o_{t_1}(v) = -\mathcal{S}(v)$, and it is considered *reversely accessed* if (ii) is changed to $-\mathcal{I}(v) = I_t^H(v)$. We sometimes add the keyword *virtually* to indicate some deviations from the definition which do not result in an altered behavior of H.

Definition 6. *A tuple $K = (H = (V^H, E^H), \mathcal{I}(\cdot), \mathcal{R}^R, \mathcal{R}^R, \mathcal{S}(\cdot))$ is a proper counter with convergence time c and supply edge number e if and only if correct access and correct initialization from t_1 to t_2 imply that the following statements hold.*

(i) $e = \sum_{\{v \in V^K | \mathcal{I}(v) \cdot \mathcal{S}(v) > 0\}} \mathcal{I}(v) \cdot \mathcal{S}(v) = \sum_{\{v \in V^K | \mathcal{I}(v) \cdot \mathcal{S}(v) < 0\}} -\mathcal{I}(v) \cdot \mathcal{S}(v)$
(ii) $\lceil \sqrt{e+2} \rceil + 1 \geq \max_{v \in V^K} |\mathcal{I}(v)|$
(iii) The vertices $\mathcal{R}^X$ are of opinion X for all $t \leq min\{t_1 + c - 1, t_2 + 1\}$
(iv) For all vertices $v \in V^H$ and all $t_1 + c \leq t \leq t_2 + 1$ the following is true $o_t(v) = -o_{t_1}(v)$.

The edge supply number corresponds to the number of black edges in Figure 2 and (i) satisfied if it is the same for blue and red. We will need condition (ii) to make sure that we do not produce a multigraph, (iii) indicates that $\mathcal{R}^X$ only change their opinion after l rounds when the graph has finished running, and (iv) makes it possible to run the counter again with reverse opinions. Note that if a counter is correctly initialized and reversely accessed, or if it is reversely initialized and correctly accessed from t_1 to t_2, $o_t(v)$ will be time-constant from t_1 to t_2 because of (iv).

2-Path graph The 2-Path graph counter will be the base case of our final, recursively defined graph. It consists just of two simple paths one of which is initialized with R and the other with B. If accessed correctly, the following will happen on both paths simultaneously. All vertices will change their opinion in the order in which they occur on their respective paths.

Definition 7. *A 2-Path graph is defined as* $P_2(l) = (H = (V^H, E^H), \mathcal{I}(\cdot), \mathcal{R}^R, \mathcal{R}^B, \mathcal{S}(\cdot))$ *where*

$$\begin{aligned}
V^H &= \{v_i^R \mid 0 \leq i < l\} \cup \{v_i^B \mid 0 \leq i < l\} \\
E^H &= \{\{v_i^R, v_{i+1}^R\} \mid 0 \leq i < l-1\} \cup \{\{v_i^B, v_{i+1}^B\} \mid 0 \leq i < l-1\} \\
\mathcal{I}(v) &= \begin{cases} -2 & \text{if } v = v_0^X \\ -1 & \text{if } v \in \{v_i^X \mid 1 \leq i < l-1\} \\ 0 & \text{otherwise} \end{cases} \\
\mathcal{R}^X &= v_{l-1}^X \\
\mathcal{S}(v_i^X) &= X
\end{aligned}$$

Lemma 4. *A 2-Path graph $P_2(l)$ is a valid counter with $n = 2l$ vertices, convergence time $c = l$ and supply edge number $e = l$.*

Proof. We will have to prove the conditions in definition 6 under the assumption that the condition in the definition 5.

The conditions (i) and (ii) are trivially true. For (iii) note that from condition (ii) of definition 5 and the definition of $\mathcal{I}(\cdot)$, it follows that the vertices in $\{v_i^X \mid 1 \leq i < l-1\}$ all have one more outside neighbor of the opinion $-X$ than of the opinion X, the vertices v_0^X has two more outside neighbors of the opinion $(-X)$ than of the opinion X, and v_{l-1}^X has the same number of outside neighbors of the opinion $(-X)$ as of the opinion X. This means that the vertices v_0^X will change its opinion to $(-X)$ in the first round, and cause v_1^X to turn in the second and so forth. In other words, $o_t(v_i^X)$ will be X for all $t \leq i$ and $(-X)$ ever after. Therefore, $o_t(\mathcal{R}^X) = o_t(v_{l-1}^X)$ is equal to X for all $t \leq l-1$, and condition (iii) of definition 6 is satisfied.

Also note that at time l all vertices will have turned and are not going to reverse back to their original position therefore satisfying condition (iv). □

Repeater A repeater is a function that takes a counter and uses transistors to repeatedly run that counter and so produce a counter with much higher convergence time at the expense of an only slightly increased number of vertices. However in addition to what was suggested in section 4 we need two new vertices $\mathcal{R}^R, \mathcal{R}^B$ to indicate when the graph has reached a stable state as displayed in Figure 9. On first reading, the reader is advised to have a look at the definition of a repeater and the content of the Lemmas 6 and 9, and then skip to section A.3.

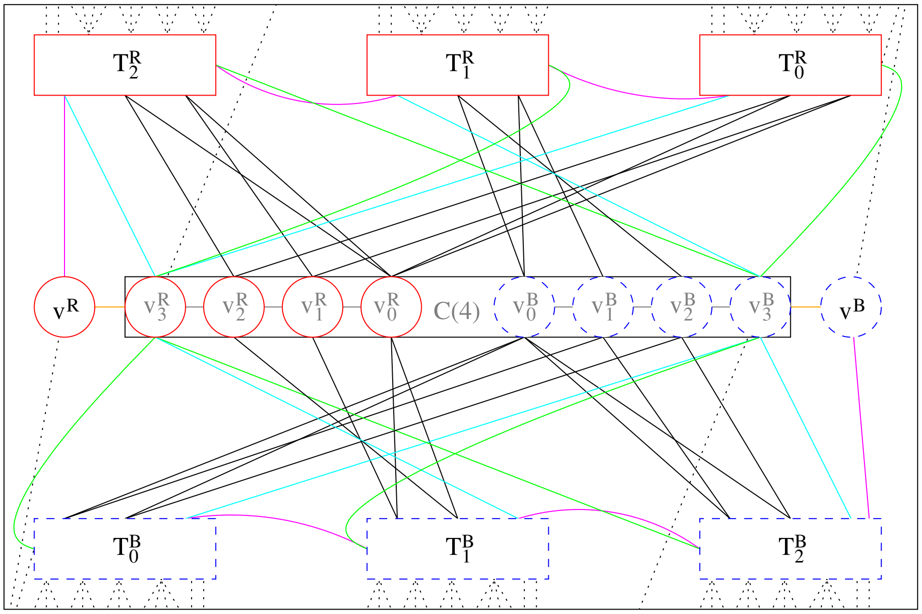

Fig. 9: Full repeater graph $R(1, P_2(4))$ with a $P_2(4)$ as graph which is repeatedly run and six transistors to run $P_2(4)$ three times.

Definition 8. *A repeater is a function* R *which when it is given a number* j *and a counter* $\tilde{K} = (\tilde{H} = (\tilde{V}^H, \tilde{E}^H), \tilde{\mathcal{I}}(\cdot), \tilde{\mathcal{R}}^R, \tilde{\mathcal{R}}^B, \tilde{\mathcal{S}}(\cdot))$ *with convergence time* $\tilde{c}$*,* $\tilde{n}$ *vertices, and supply* *edge number $\tilde{e}$, produces a tuple $R(\tilde{K}, j) = (H = (V^H, E^H), \mathcal{I}(\cdot), \mathcal{R}^R, \mathcal{R}^B, \mathcal{S}(\cdot))$ where*

$$T_i^X = (V_i^X, E_i^X) \text{ for } 0 \le i \le 2j, X \in \{R, B\} \text{ are instances of transistors } T(s)$$

$$V^H = \left(\bigcup_{X \in \{R,B\}, i=0}^{j-1} V_i^X \right) \cup \tilde{V}^H \cup v^R \cup v^B$$

$$E^H = \left(\bigcup_{X \in \{R,B\}, i=0}^{j-1} E_i^X \right) \cup \tilde{E}^H \cup E^1 \cup E^2 \cup E^3 \cup E^4 \cup E^5$$

$$E^1 = \{\{\tilde{\mathcal{R}}^X, \mathcal{B}^1(T_{2i}^{-X})\} \mid 0 \le i \le 2j\} \cup \{\tilde{\mathcal{R}}^X, \mathcal{B}^1(T_{2i+1}^X)\} \mid 0 \le i \le 2j-1\} \text{ (green in Figure 9)}$$

$$E^2 = \{\{\tilde{\mathcal{R}}^X, \mathcal{R}^X\} | X \in \{R, B\}\} \text{ (orange in Figure 9)}$$

$$\mathcal{I}(v) = \begin{cases} -2 & \text{if } v = \mathcal{B}^1(T_0^X) \\ -1 & \text{if } v = \mathcal{B}^1(T_i^X) \text{ for some } 1 \le i \le 2j \\ -1 & \text{if } v = \mathcal{B}^2(T_i^X) \text{ for some } 0 \le i \le 2j \\ -(s+1) & \text{if } v = \mathcal{B}^3(T_i^X) \text{ for some } 0 \le i \le 2j \\ -s & \text{if } v \in \mathcal{E}(T_i^X) \\ 1 & \text{if } v = \mathcal{R}^X \\ -1 & \text{if } v = \tilde{\mathcal{R}}^X \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{R}^R = v^R$$

$$\mathcal{R}^B = v^B$$

$$\mathcal{S}(v) = \begin{cases} \tilde{\mathcal{S}}(v) & \text{if } v \in \tilde{V^T} \\ X & \text{if } v \in V_i^X \\ X & \text{if } v = \mathcal{R}^X \end{cases}$$

$$s = \lceil \sqrt{\tilde{e}+2} \rceil + 1$$

E^3 (black in Figure 9) consists the following edges. For every vertex $v \in \tilde{V}^H$ with $k = |\tilde{\mathcal{I}}(v)|$ and with $X = sign(\tilde{\mathcal{I}}(v)\mathcal{S}(v))$, there are edges from v to k different emitter vertices of every T_{2i}^{-X} with $0 \le i \le j$ and of every T_{2i+1}^X with $0 \le i \le j-1$.

E^4 (magenta in Figure 9) consists of the following edges. For every $0 \ge i \ge 2j-1$ and X, there is an edge from $\mathcal{B}^1(T_{i+1}^X)$ to an emitter vertex of T_i^X and there is one edge from an emitter vertex of T_{2j}^X to $\mathcal{R}^X$.

E^5 (cyan in Figure 9) consists of the following edges. For all X, there is an edge from $\tilde{\mathcal{R}}^X$ to an emitter vertex of every T_{2i}^X with $0 \le i \le j$ and of every T_{2i+1}^{-X} with $0 \le i \le j-1$.

Lemma 5. *The repeater $R(\tilde{K}, j)$ does indeed exist such that the emitter vertices of T_i are connected to no more than $(s-1)$ vertices outside T_i.*

Proof. We have to show that there are E^3, E^4 and E^5 such that the emitter vertices of the transistors do not have too many outside edges. Every transistor's emitter vertices have $\tilde{e}$ edges leaving the transistor in E^3 because of condition (i) of definition 5, one edge in E^4 and one edge in E^5. This is a total of $\tilde{e}+2$ necessary edges. A transistor $T(s)$ has s collector vertices each of which should have no more than $(s-1)$ outside edges where $s = \lceil \sqrt{\tilde{e}+2} \rceil + 1$. This makes a total of more than possible $\tilde{e}+2$ edges.

$$\begin{aligned} s(s-1) &= (\lceil\sqrt{\tilde{e}+2}\rceil + 1)(\lceil\sqrt{\tilde{e}+2}\rceil + 1 - 1) \\ &> \lceil\sqrt{\tilde{e}+2}\rceil^2 \\ &\geq \tilde{e}+2 \end{aligned}$$

Additionally condition (ii) of definition 5 shows that $|\tilde{\mathcal{I}}(v)| \leq s$. This is necessary because we could otherwise only realize the graph if we were allowed multigraphs. □

Lemma 6. *The repeater $R(\tilde{K}, j)$ has $n = 2(2j+1)(2s+1) + \tilde{n} + 2$ vertices and supply edge number $e = (2j+1)(s^2+s+3)+3$ where $s = \lceil\sqrt{e+2}\rceil + 1$.*

Proof. To obtain the number of vertices we add the number of vertices in a transistor times the number of transistors to the number in the counter K and two vertices $\mathcal{R}^X$.

$$\begin{aligned} |V^H| &= 2(2j+1)|T(s)| + |\tilde{V}^H| + 2 \\ &= 2(2j+1)(2s+3) + n + 2 \end{aligned}$$

To obtain the supply edge number we just the go through the cases in the definition of $\mathcal{S}(\cdot)$ and add them up.

$$\begin{aligned} e &= 1\cdot 2 + 2j\cdot 1 + (2j+1)\cdot 1 + (2j+1)\cdot(s+1) + (2j+1)s\cdot s + 1\cdot 1 + 1\cdot 1 \\ &= 2(2j+1) + (2j+1)(s+1) + (2j+1)s^2 + 3 \\ &= (2j+1)(s^2+s+3)+3 \end{aligned}$$

□

Lemma 7. *For all $0 \leq i \leq 2j$ and all X, T_i^X is accessed correctly.*

Proof. Condition (i) of definition 4 is trivially true. The collector vertices as well as $\mathcal{B}^2$ and $\mathcal{B}^3$ of T_i have no edges to other parts of H so it holds that $I_t^{T_i}(v) = I_t^H(v) = \mathcal{I}(v)$ for all $v \in \mathcal{C}(T_i) \cup \{\mathcal{B}^2(T_i)\} \cup \{\mathcal{B}^3(T_i)\}$. Therefore, (ii),(v) and (vi) are fulfilled.

Condition (iii) of is fulfilled because of Lemma 5.

For condition (iv) we distinguish between $i=0$ and $i \neq 0$. If i is equal to 0, then $\mathcal{B}^1(T_i^X)$ has 1 edge coming from outside V_i^X but still inside H and an outside influence of $I^H(\mathcal{B}^1(T_i^X)) = -2$. If i is not equal to 0, then $\mathcal{B}^1(T_i^X)$ has 2 edges coming from outside V_i^X but still inside H and an outside influence of $I^H(\mathcal{B}^1(T_i^X)) = -1$. In both cases, the resulting $I^{T_i^X}(\mathcal{B}^1(T_i^X))$ will satisfy (iv). □

Lemma 8. *The transistors flip in order. That is, the following statements must hold.*

(i) If $t^f(T_i^X) = t$ there must be a $t' \leq t-4$ such that $t^f(T_{i-1}^X) = t'$ for all $1 \leq i \leq 2j$.
(ii) If $O_t^{\mathcal{R}^X}(\mathcal{R}^X) = -X$ there must be a $t' \leq t-4$ such that $t^f(T_{2j}^X) = t'$

Proof. The vertex $\mathcal{R}^X$ has total influence $I_t^v(\mathcal{R}^X) = I_t^H(\mathcal{R}^X) + o_t(\tilde{\mathcal{R}}^X) + o_t(\mathcal{E}(T_{2j}^X))$ which is equal to $1 + X\cdot(o_t(\tilde{\mathcal{R}}^X) + o_t(\mathcal{E}(T_{2j}^X)))$. So $\mathcal{R}^X$ can only change its opinion to $(-X)$ if both $o_t(\tilde{\mathcal{R}}^X)$ and $o_t(\mathcal{E}(T_{2j}^X))$ are $(-X)$. Similarly $I_t^{T_i^X}(\mathcal{B}(T_i^X))$ which can be written as $I_t^H(\mathcal{B}(T_i^X)) + X(o_t(\mathcal{E}(T_{i-1}^X)) + o_t(\tilde{\mathcal{R}})) = -1 + X(o_t(\mathcal{E}(T_{i-1}^X)) + o_t(\tilde{\mathcal{R}}))$ can only be -3 if $\mathcal{E}(T_{i-1}^X)$ is $-X$ for all $1 \leq i \leq 2j$. This together with statements (i) and (ii) of Lemma 3 we get the required statements. □

Definition 9. *We define* ${}^1I^{\tilde{H}}(\cdot)$, ${}^2I^{\tilde{H}}(\cdot)$, ${}^3I^{\tilde{H}}(\cdot)$ *and* ${}^5I^{\tilde{H}}(\cdot)$ *to be the outside influence exerted on vertices in* $\tilde{H}$ *by* E^1, E^2, E^3 *and* E^5 *respectively.*

Note that $I_t^{\tilde{H}}(v) = {}^1I_t^{\tilde{H}}(v) + {}^2I_t^{\tilde{H}}(v) + {}^3I_t^{\tilde{H}}(v) + {}^5I_t^{\tilde{H}}(v) + I_t^H(v)$. The edges E^4 are intentionally left out since they do not have endpoints in $\tilde{H}$.

Lemma 9. *$R(j, \tilde{K})$ is indeed a counter of convergence time $c = (2j+1)(\tilde{c}+4)+1$.*

Proof. So assume that the constant outside influence of all vertices is given by $I^H(\cdot) = \mathcal{I}^H(\cdot)$. We define c to be the smallest t with $o_t(\mathcal{R}^R) = B$ or $o_t(\mathcal{R}^B) = R$. By Lemma 8 all transistors have flipped at time c. For all $t < c$ and all vertices $v \in \tilde{H}$, the outside influences of ${}^2I_t^{\tilde{H}}(v)$ and of $I_t^H(v)$ cancel out each other so we need only care about ${}^1I^{\tilde{H}}()$, ${}^3I^{\tilde{H}}()$ and ${}^5I^{\tilde{H}}()$ until we reach time c.

To get a induction hypothesis we prove the following stronger statement: $t^f(T^{2l}) = 2l(\tilde{c}+4)$ and $o_{2l(\tilde{c}+4)}(v) = \mathcal{S}(v)$ for all $0 \leq l \leq j$ and for all vertices v in $\tilde{H}$.

The base case for $l = 0$ is trivial. For the induction step $l \to l+1$ assume $t^f(T^{2l}) = 2l(\tilde{c}+4)$ and $o_{2l}(v) = \mathcal{S}(v)$ is true for all vertices v in $\tilde{H}$. We will in the following look at how the graph behaves in the following intervals $[2l(\tilde{c}+4), 2l(\tilde{c}+4)+3]$, $[2l(\tilde{c}+4)+4, (2l+1)(\tilde{c}+4)-1]$, $[(2l+1)(\tilde{c}+4), (2l+1)(\tilde{c}+4)+3]$ and $[(2l+1)(\tilde{c}+4)+4, (2l+2)(\tilde{c}+4)-1]$. I.e., we take the behavior during the interval $[a, b]$ to mean the influences exerted in $[a, b]$ and their outcomes in $[a+1, b+1]$.

Interval $[2l(\tilde{c}+4), 2l(\tilde{c}+4)+3]$. We will show that $\tilde{K}$ is correctly initialized and virtually reversely accessed from $2l(\tilde{c}+4)$ to $2l(\tilde{c}+4)+3$. Therefore and because of (iii) in definition 6, $o_{2l(\tilde{c}+4)+4}(\cdot)$ will still be equal to $\mathcal{S}(\cdot)$.

Let us start by considering ${}^1I^{\tilde{H}}(\cdot) + {}^5I^{\tilde{H}}(\cdot)$. Using Lemma 8 we can deduce the following statement. All transistors T_i^X with $i < 2l$ have already switched completely by $2l(\tilde{c}+4)$ (they are in the region specified by (iii) of Lemma 3, similarly all such transistors with $i > 2l$ can only start switching after $i(\tilde{c}+4)+3$ (they are in the region specified by (i) of Lemma 3. So the contribution of T_i^Y to ${}^1I^{\tilde{H}}()$ is canceled out by the contribution of T_i^{-Y} to ${}^5I^{\tilde{H}}()$ for all $i \neq 2l$ and vice versa. And for T_{2l}^X we know that $o_i(\mathcal{E}(T_{2j}^X)) = X$ for $i \leq 2l(\tilde{c}_4)+3$ from (i), (ii) of Lemma 3. Therefore, ${}^1I_i^{\tilde{H}}(v) + {}^5I_i^{\tilde{H}}(v)$ must be non-negative for $\tilde{\mathcal{R}}^X$ and zero for all other vertices.

${}^3I_i^{\tilde{H}}(\cdot)$ will be exactly $-\mathcal{I}(\cdot)$ for $2l(\tilde{c}+4) \leq i \leq 2l(\tilde{c}+4)+3$ because the transistors T_i^Y with an even $i \neq 2l$ cancel out each other and those with $i = 2l$ will have $o_t(\mathcal{E}(T_i^X)) = X$ for $t \leq 2l(\tilde{c}+4)+3$ so T_i^X will exactly contribute the required outside influence of $-\mathcal{I}(\cdot)$. So $I_t^T(v) = {}^1I_t^{\tilde{H}}(v) + {}^3I_t^{\tilde{H}}(v) + {}^5I_t^{\tilde{H}}(v)$ will be $-\mathcal{I}(v)$ for all v except for $\mathcal{R}^R$ and $\mathcal{R}^B$. However $\mathcal{R}^X$ is supposed to stick with $o(\mathcal{R}^X) = X$, in case of correct initialization and reverse access and the deviation introduced by ${}^1I^{\tilde{H}}() + {}^5I^{\tilde{H}}()$ further encourages them to do so. Therefore $\tilde{K}$ is a correctly initialized and virtually reversely accessed counter.

Interval $[2l(\tilde{c}+4)+4, (2l+1)(\tilde{c}+4)-1]$. We will use a proof of induction over k to show that $\tilde{K}$ is accessed and initialized correctly from $2l(\tilde{c}+4)+4$ to $2l(\tilde{c}+4)+4+k$ and that $t^f(T_{2l+1}^X) \geq 2l(\tilde{c}+4)+k$ for all $k < \tilde{c}$. Using (iv) of definition 6 the induction statement for k set to $\tilde{c}-1$ will imply that $o_{(2l+1)(\tilde{c}+4)}(\cdot) = -\tilde{\mathcal{S}}(\cdot)$. This in turn will imply that $t^f(T_{2l+1}^X) = (2l+1)(\tilde{c}+4)$.

Because of Lemma 8 and because $o_{2l(\tilde{c}+4)}(\tilde{\mathcal{R}}^X) = X$, $t^f(T^e_{2l+1})$ must be bigger than $2l(\tilde{c}+4)+4$. Therefore, of all transistors each one will be of uniform opinion at $2l(\tilde{c}+4)$, and therefore ${}^1I^{\tilde{H}}_{2l+4}() + {}^5I^{\tilde{H}}_{2l+4}()$ will be 0. ${}^3I^{\tilde{H}}_{2l+4}(v)$ will be exactly $\mathcal{I}(v)$. This is because for the same reason as in the previous interval all transistors T^X_i with $i \neq 2l$ cancel out the transistors $i = 2l$ have already flipped completely. Hence, $\tilde{K}$ is accessed and initialized correctly from $2l(\tilde{c}+4)+4$ to $2l(\tilde{c}+4)+4$. This covers the base case of our induction.

Now assuming $\tilde{K}$ is accessed and initialized correctly from $2l(\tilde{c}+4)+4$ to $2l(\tilde{c}+4)+4+k$ and $t^f(T^X_{2l+1}) \geq 2l(\tilde{c}+4)+k$ for some $k < \tilde{c}$. Because of the first assumption and because $k+1 < \tilde{c}$ we can apply (iii) of definition 6. Hence $o_{2l(\tilde{c}+4)+k}(\tilde{\mathcal{R}}^X) = X$ will still be true and as a direct consequence $t^f(T^X_{2l+1}) \geq 2l(\tilde{c}+4)+k+1$ and we have proven the first part of our induction statement. Now since $t^f(T^X_{2l+1}) \geq 2l(\tilde{c}+4)+k+1$ T^X_{2l+1} is still uniformly of the opinion X at $2l(\tilde{c}+4)+k+1$, so $I^{\tilde{H}}_{2l(\tilde{c}+4)+k+1}(\cdot)$ is still equal to $\mathcal{I}(\cdot)$. This also means that $\tilde{K}$ is still correctly accessed and therefore proves our induction statement.

Interval $[(2l+1)(\tilde{c}+4), (2l+1)(\tilde{c}+4)+3]$ The graph $\tilde{K}$ is reversely initialized and virtually correctly accessed from $(2l+1)(\tilde{c}+4)$ to $(2l+1)(\tilde{c}+4)+3$. We know from the last interval that $\tilde{K}$ is reversely initialized and that $T^X_{(2l+1)} = (2l+1)(\tilde{c}+4)$ The proof of correct access is completely analogous to the first interval. Therefore and because of (iii) in definition 6, $o_{2l(\tilde{c}+4)+4}(\cdot)$ will still be equal to $-\mathcal{S}(\cdot)$.

Interval $[(2l+1)(\tilde{c}+4)+4, (2l+2)(\tilde{c}+4)-1]$ The graph $\tilde{K}$ is accessed and initialized reversely from $(2l+1)(\tilde{c}+4)+4$ to $(2l+2)(\tilde{c}+4)-1$ and $t^f(T^X_{2l+2}) = 2l(\tilde{c}+4)$. We know from the last interval that $\tilde{K}$ is reversely initialized the rest of the proof is completely analogous to the second interval. Therefore $o_{(2l+2)(\tilde{c}+4)}(\cdot) = \tilde{\mathcal{S}}(\cdot)$.

This concludes the proof of the induction step. Now we only need to show that we can indeed derive the Lemma from this induction result. At time $2j(\tilde{c}+4)$ we basically have the same case as at the beginning of the first interval and using the same technique as in the first and second interval we can deduce that the first time t with $I^{\tilde{H}}_t(\mathcal{R}^X) < 0$ is $(2j+1)(\tilde{c}+4)$ (this proves (iii) of definition 6). We can in the same manner prove that $o_{(2j+1)(\tilde{c}+4)}(v) = -\mathcal{S}(v)$ for all $v \in \tilde{V}^H$. And since all transistors have already flipped completely and because $\tilde{K}$ is reversely initialized and correctly accessed from $(2j+1)(\tilde{c}+4)$ to infinity (iv) is also fulfilled. The conditions (i) are true because of Lemma 6 and finally (ii) is true since

$$\begin{aligned}
\lceil\sqrt{e+2}\rceil + 1 &= \lceil\sqrt{(2j+1)(s^2+s+3)+3+2}\rceil + 1 \\
&\geq \lceil\sqrt{s^2}\rceil + 1 \\
&= s+1 \\
&= \max_{v \in V^H} \mathcal{I}(v).
\end{aligned}$$

□

A.3 Putting it all together

Lemma 10. *For every $l \geq 1$ and $h \geq 0$, there is a counter K with $n \leq (h+1)54\left(1+\frac{80}{\sqrt{l}}\right)^h l$ vertices, supply edge number $e \leq l^{2-\frac{1}{2^h}} \cdot \left(1+\frac{80}{\sqrt{l}}\right)^h$ and convergence time $c \geq l^{2-\frac{1}{2^h}}$.*

Proof. We prove this by induction over h. For $h = 0$ K is trivially given by $P_2(l)$. Now given a counter $\tilde{K}$ with $\tilde{n} \leq (h+1)54\left(1+\frac{80}{\sqrt{l}}\right)^h l$ vertices, convergence time $\tilde{c} \geq l^{2-\frac{1}{2^h}}$ and supply edge number $\tilde{e} \leq l^{2-\frac{1}{2^h}} \cdot \left(1+\frac{80}{\sqrt{l}}\right)^h$ we construct $K = R(\tilde{K}, \frac{1}{2}\lfloor l^{\frac{1}{2^{h+1}}}\rfloor)$. By Lemma 9 the convergence time c is at least $(2\frac{1}{2}\lfloor l^{\frac{1}{2^{h+1}}}\rfloor + 1)(l^{2-\frac{1}{2^h}} + 4)$ which is in turn at least $l^{2-\frac{1}{2^{h+1}}}$. To prove the bound on the supply edge number, we first show a bound for the transistor size s by using the definition of R. The required bound for the supply edge number of K can be deduced using Lemma 6.

$$\begin{aligned}
s &= \lceil\sqrt{\tilde{e}+2}\rceil + 1 && (1)\\
&\leq \sqrt{l^{2-\frac{1}{2^h}} \cdot \left(1+\frac{80}{\sqrt{l}}\right)^h + 2} + 2 && (2)\\
&\leq \sqrt{\left(1+\frac{2}{l}\right) l^{2-\frac{1}{2^h}} \cdot \left(1+\frac{80}{\sqrt{l}}\right)^h} + 2 && (3)\\
&\leq \left(1+\frac{2}{l}\right)\sqrt{l^{2-\frac{1}{2^h}} \cdot \left(1+\frac{80}{\sqrt{l}}\right)^h} + 2 && (4)\\
&\leq \left(1+\frac{2}{l}+\frac{2}{\sqrt{l}}\right)\sqrt{l^{2-\frac{1}{2^h}} \cdot \left(1+\frac{80}{\sqrt{l}}\right)^h} && (5)\\
&\leq \left(1+\frac{4}{\sqrt{l}}\right)\sqrt{l^{2-\frac{1}{2^h}} \cdot \left(1+\frac{80}{\sqrt{l}}\right)^h} && (6)
\end{aligned}$$

$$\begin{aligned}
e &= (2j+1)(s^2+s+3)+3 \qquad\qquad \text{where } j = \frac{1}{2}\lfloor l^{\frac{1}{2^{h+1}}}\rfloor\\
&\leq \left(1+\frac{1}{l}\right) \cdot l^{\frac{1}{2^{h+1}}}(s^2+s+3)+3\\
&\leq \left(1+\frac{1}{l}\right) \cdot l^{\frac{1}{2^{h+1}}}\left(\left(s+\frac{1}{2}\right)^2+\frac{11}{4}\right)+3\\
&\leq \left(1+\frac{1}{l}\right) \cdot l^{\frac{1}{2^{h+1}}}\left(\left(\left(1+\frac{4}{\sqrt{l}}\right)\sqrt{l^{2-\frac{1}{2^h}} \cdot \left(1+\frac{80}{\sqrt{l}}\right)^h}+\frac{1}{2}\right)^2+\frac{11}{4}\right)+3\\
&\leq \left(1+\frac{1}{l}\right) \cdot l^{\frac{1}{2^{h+1}}}\left(\left(\left(1+\frac{5}{\sqrt{l}}\right)\sqrt{l^{2-\frac{1}{2^h}} \cdot \left(1+\frac{80}{\sqrt{l}}\right)^h}\right)^2+\frac{11}{4}\right)+3\\
&\leq \left(1+\frac{1}{l}\right) \cdot l^{\frac{1}{2^{h+1}}}\left(\left(1+\frac{10}{\sqrt{l}}+\frac{25}{l}\right) l^{2-\frac{1}{2^h}} \cdot \left(1+\frac{80}{\sqrt{l}}\right)^h+\frac{11}{4}\right)+3\\
&\leq \left(1+\frac{1}{l}\right) \cdot l^{\frac{1}{2^{h+1}}}\left(1+\frac{38}{\sqrt{l}}\right) l^{2-\frac{1}{2^h}} \cdot \left(1+\frac{80}{\sqrt{l}}\right)^h+3\\
&\leq \left(1+\frac{77}{\sqrt{l}}\right) \cdot l^{2-\frac{1}{2^{h+1}}} \cdot \left(1+\frac{80}{\sqrt{l}}\right)^h+3\\
&= \left(1+\frac{80}{\sqrt{l}}\right)^{h+1} l^{2-\frac{1}{2^{h+1}}}
\end{aligned}$$

When contracting terms in (3) and (5), we use the fact that $l^{2-\frac{1}{2^h}} \geq l$ and $\left(1+\frac{80}{\sqrt{l}}\right)^h \geq 1$. Using the same lemma, also gives the needed bound for the number of vertices n.

$$
\begin{aligned}
n &= 2(2j+1)(2s+3) + \tilde{n} + 2 \qquad\qquad \text{where } \tilde{n} = (h+1)54\left(1+\frac{80}{\sqrt{l}}\right)^h l \\
&\leq 2\left(1+\frac{1}{l}\right) l^{\frac{1}{2^{h+1}}} (2s+3) + \tilde{n} + 2 \\
&\leq \left(2+\frac{2}{l}\right) l^{\frac{1}{2^{h+1}}} \left(2\left(1+\frac{4}{\sqrt{l}}\right)\sqrt{l^{2-\frac{1}{2^h}} \cdot \left(1+\frac{80}{\sqrt{l}}\right)^h} + 3\right) + \tilde{n} + 2 \\
&\leq \left(2+\frac{2}{l}\right) l^{\frac{1}{2^{h+1}}} \left(2+\frac{11}{\sqrt{l}}\right)\sqrt{l^{2-\frac{1}{2^h}} \cdot \left(1+\frac{80}{\sqrt{l}}\right)^h} + \tilde{n} + 2 \\
&\leq \left(4+\frac{48}{\sqrt{l}}\right) l^{\frac{1}{2^{h+1}}} \cdot l^{1-\frac{1}{2^{h+1}}} \left(1+\frac{80}{\sqrt{l}}\right)^{\frac{h}{2}} + \tilde{n} + 2 \\
&\leq \left(4+\frac{50}{\sqrt{l}}\right)\left(1+\frac{80}{\sqrt{l}}\right)^{\frac{h}{2}} l + \tilde{n} \\
&\leq 54\left(1+\frac{80}{\sqrt{l}}\right)^{\frac{h}{2}} l + (h+1)54\left(1+\frac{80}{\sqrt{l}}\right)^h l \\
&\leq (h+2)54\left(1+\frac{80}{\sqrt{l}}\right)^h l \\
&\leq (h+2)54\left(1+\frac{80}{\sqrt{l}}\right)^{h+1} l
\end{aligned}
$$

□

We need one more additional tool from mathematics to proof our final Theorem 3.

Lemma 11. *For every $x > 0$ and all $n > 0$ the following inequality holds $\left(1+\frac{x}{n}\right)^n \leq e^x$*

Proof. It is well known that the sequence $s_n(x) = \left(1+\frac{x}{n}\right)^n$ converges to e^x as n goes to infinity. So we need only proof that $s_n(x)$ is non-decreasing in n. We achieve this by showing that all coefficients in the power series $s_n(x) = \sum_{k=0}^{\infty} c_n^k x^k = \sum_{k=0}^{\infty} \binom{n}{k} \left(\frac{x}{n}\right)^k$ are non decreasing.

$$
\begin{aligned}
\frac{c_{n+1}^k}{c_n^k} &= \frac{\binom{n+1}{k} \frac{1}{(n+1)^k}}{\binom{n}{k} \frac{1}{n^k}} && (7)\\
&= \frac{\frac{(n+1)!}{k!(n+1-k)!} \frac{1}{(n+1)^k}}{\frac{n!}{k!(n-k)!} \frac{1}{n^k}} && (8)\\
&= \frac{\frac{1}{n+1-k} \frac{1}{(n+1)^{k-1}}}{\frac{1}{n^k}} && (9)\\
&= \frac{n^k}{(n+1)^{k-1}(n-(k-1))} && (10)\\
&\geq \frac{n^k}{n^k} && (11)\\
&= 1 && (12)
\end{aligned}
$$

In (10), the arithmetic mean of the factors in the denominator (numerator respectively) is n. Since the geometric mean of positive numbers can never be bigger than the arithmetic mean, the geometric mean of the factors of the denominator has to be $(\leq n)$ and the product $(\leq n^k)$. □

Now we combine Lemma 10 and Lemma 11 to proof Theorem 3.

Proof. If we select h in Lemma 10 to be $\lfloor \log\log l \rfloor$ we can get a counter with the following dimensions for every l.

$$
\begin{aligned}
n &\le (\lfloor \log\log l \rfloor + 1) 54 \left(1 + \frac{80}{\sqrt{l}}\right)^{\lfloor \log\log l \rfloor} l && (13)\\
&\le 108 \log\log l \left(1 + \frac{80}{\lfloor \sqrt{l} \rfloor}\right)^{\lfloor \log\log l \rfloor} l && (14)\\
&\le 108 \log\log l \left(1 + \frac{80}{\lfloor \sqrt{l} \rfloor}\right)^{\lfloor \sqrt{l} \rfloor} l && (15)\\
&\le 108 \cdot e^{80} \log\log l \cdot l && (16)\\
e &\le l^2 \cdot l^{-\frac{1}{2^{\lfloor \log\log l \rfloor}}} \cdot \left(1 + \frac{80}{\sqrt{l}}\right)^{\lfloor \log\log l \rfloor} && (17)\\
&\le l^2 \cdot l^{-\frac{1}{\log l}} \cdot e^{80} && (18)\\
&= \frac{1}{2} l^2 \cdot e^{80} && (19)\\
c &\ge l^2 \cdot l^{-\frac{1}{2^{\lfloor \log\log l \rfloor}}} && (20)\\
&\ge l^2 \cdot l^{-\frac{1}{2^{(\log\log l) - 1}}} && (21)\\
&= l^2 \cdot l^{-\frac{2}{\log l}} && (22)\\
&= l^2 \cdot \left(l^{-\frac{1}{\log l}}\right)^2 && (23)\\
&= \left(\frac{1}{2}\right)^2 l^2 && (24)\\
&= \frac{1}{4} l^2 && (25)
\end{aligned}
$$

(16) and (18) are true because of Lemma 11, and (19) and (24) are true because it holds that $\log\left(l^{-\frac{1}{\log l}}\right) = -\frac{1}{\log l} \log l = -1 = \log \frac{1}{2}$. We can also run this counter by creating a red and a blue clique of size $\lceil \sqrt{e} \rceil + 1$ and then connecting the vertices in the counter to vertices in the cliques according to $\mathcal{I}(\cdot)$. Since $\sqrt{e} = \mathcal{O}(n)$ this increases the number of vertices only by a constant fraction. So our final network has $n = \mathcal{O}(l \cdot \log\log l)$ vertices and a convergence time of $\Omega(l^2) = \Omega\left(\frac{n^2}{(\log\log l)^2}\right) \ge \Omega\left(\frac{n^2}{(\log\log n)^2}\right)$. ◻